Against the Topologists: Essay Review of *New Foundations for Physical Geometry*

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Tim Maudlin begins *New Foundations for Physical Geometry*, his first book in a planned pair, with a headlong charge: "The thesis of these books is both simple and audacious. It is so simple that the basic claims can be reduced to two sentences. First: the most fundamental geometrical structure that organizes physical points into a space is the line. Second: what endows spacetime with its geometry is time" (1). Although he hints at applications forthcoming in the second volume to Newtonian and Relativistic spacetime physics, including the problem of black hole evaporation (25), this volume concentrates on an extended development of the mathematics of his *Theory of Linear Structures*.

In fact, Maudlin is interested in contrasting his theory as an alternative to (1) and rival of (112) the field of mathematics called *topology*, which concerns the features of spaces invariant under continuous deformation. Such deformations preserve properties like two points being "connected" in the space but not properties like the distance between them (if this is well-defined). This is the sense in which Maudlin is attempting to construct a new foundation for "submetrical" geometry: he believes topology has given incorrect analyses and definitions for what he takes to be pre-existing geometric concepts like "continuity," which the Theory of Linear Structures is attempting to rectify. Accordingly, he insists on using the same words for many concepts and properties defined in his theory as those in topology—e.g., continuity, connectedness, openness, closedness, and convergence—despite them having non-equivalent definitions. This is because "the Theory of Linear Structures has the same right as standard topology to these terms" (26), which he believes denote fixed, if informal, concepts. Maudlin's exposition is technically accessible, presupposing only elementary set theory and order theory, and although familiarity with elementary topology would be helpful, the simple exercises that follow most chapters are useful checks of one's understanding.

After a discussion of his motivations for the project—again, to "offer a better understanding of geometrical structure, and allow for definitions that more closely capture the intuitive notions we are trying to explicate" (2)—and a chapter criticizing standard topology, the bulk of the book concentrates on the mathematics of the Theory of Linear Structures, often explicitly in comparison with topology. Both standard topology and Maudlin's theory begin with an arbitrary set of elements S to which extra structure is added. For topology, this structure is often the collection of open sets.[1] By contrast, Maudlin takes as his basic structure on S the set of (directed) lines Λ, which is a non-empty set of subsets of S, each element λ of which is endowed with a linear order $>_\lambda$.[2] The lines of Λ are required to satisfy the following axioms:

1. If $\lambda \in \Lambda$, then λ contains at least two elements.
2. For every $\lambda \in \Lambda$, each interval of $>_\lambda$ containing at least two elements is in Λ.[3]
3. If $\lambda, \mu \in \Lambda$ have in common only a single point p that is the upper bound of $>_\lambda$ and the lower bound of $>_\mu$, then the union of their point sets with the order that agrees with $>_\lambda$ and $>_\mu$ is also in Λ, provided that no line whose points are in $(\lambda \cup \mu) - p$ has a point in both λ and μ.
4. Every $\sigma \subseteq S$ that can be endowed with a linear order $>_\sigma$ whose closed intervals containing at least two points are lines is also a line.[4]

[1] There are many equivalent axiomatizations of topology using different structures—see, for example, Willard (1970/2004, Ch. 2)—but Maudlin, in his comparisons, follows modern presentations, which begin with the austere and algebraically elegant open sets.

[2] Maudlin actually introduces undirected and directed Linear Structures separately, but the former can be canonically reconstructed from the latter (97), so this simplification does no harm.

[3] Maudlin actually uses a somewhat different axiom:

2. For each $\lambda \in \Lambda$, all and only the intervals of $>_\lambda$ are segments of λ. Any other $\mu \in \Lambda$ whose points form a subset of λ is an inverse segment of λ.

He defines a(n) (*inverse*) *segment* μ of λ as a line such that for all pairs of points p, q that λ and μ have in common, $p >_\lambda q$ if and only if $p >_\mu q$ ($q >_\mu p$). However, this definition (94) and the analogous one for undirected linear structures (60) are defective. Consider a "triangle" of points, that is, a set $S = \{p, q, r\}$, whose lines are $\lambda = \langle p, q, r \rangle$, $\mu = \langle p, r \rangle$, and their intervals and concatenations. By Maudlin's *segment* definition, μ is a segment of λ, yet by his axiom it is not a segment of λ, which is a contradiction. So this "triangle" is not a legitimate Linear Structure, yet it is almost identical to one that Maudlin considers and endorses (91). The most obvious way to repair this defect is to define a *segment* of any $\lambda \in \Lambda$ as any interval of λ containing at least two elements.

[4] Maudlin again actually has a slightly different axiom:

4. Every $\sigma \subseteq S$ which, when endowed with a linear order $>_\sigma$, is such that all and only the codirectional closed lines whose points lie in σ are closed intervals of $>_\sigma$ is a line.

He defines a line to be *closed* when it has a least and greatest element, and two lines λ and μ to be *codirectional* when they contain points p and q such that $p >_\lambda q$ and $p >_\mu q$. This axiom (96) as it stands is also defective (as is the analogous one for undirected linear structures (65)). In the first place, "the codirectional closed lines whose points lie in σ" refers to possibly infinitely many distinct classes—one for each pair of distinct points in σ—since codirectionality is not transitive relation between lines, so presumably it should be prefixed with something like "one of the classes of". But even with this, "all" is too strong a requirement. For, consider the integers $\mathbb{Z}$ with lines $\langle n, n+1, \ldots, m \rangle$ for all $n, m \in \mathbb{Z}$ with $n < m$, the line $\langle -1, 1 \rangle$, as well as all their intervals and finite concatenations. (Imagine the shape of an "A" with the legs extending off to infinity.) The closed line $\langle -1, 1 \rangle$ is codirectional with all the other lines containing $\{-1, 1\}$, but it is not an interval of $\mathbb{Z}$ with its standard linear order. Thus it is consistent with the axiom as stated that $\mathbb{Z}$ with its standard linear order is *not* a line of the Linear Structure. Yet this freedom is exactly what Maudlin wants to avoid, for he introduces axiom 4 explicitly so that the lines of discrete Linear Structures are determined completely by their minimal (2-element) lines. In amending this, it seems simplest just to eliminate the reference to codirectionality.

The first axiom requires lines to have at least two points. The second effectively requires that the intervals of lines (with respect to their linear order) are also lines. The third demands that the concatenation of two lines forms a line, provided that the result does not self-intersect or form a circle. The fourth generalizes this concatenation property to hold for arbitrarily many lines, not just finite sequences thereof.

Maudlin uses this linear order-theoretic foundation to construct his own versions of structures and properties considered for topological spaces, such as neighborhoods, open and closed sets, boundary points, connectedness, convergence, the continuity of functions, and subspaces, contrasting their features with their counterparts in standard topology. He also goes on to develop analogs of more geometric concepts and constructions, such as affine spaces, convexity, tangency, and metrical (distance) structure, before returning to analogs of concepts from (geometric) topology, including product spaces, fiber bundles, homotopy, and compactness. There are also many asides connecting and motivating his ideas with those of ancient geometers, in particular Zeno, Eudoxus, and, of course, Euclid.

While most of the book is occupied with the mathematical construction of the Theory of Linear Structures and conceptual explorations in service thereof, Maudlin is also at pains throughout to emphasize what he sees as the differences and conceptual advantages of his theory with those of standard topology, even beyond the first chapter, entitled "Topology and Its Shortcomings." In the course of this development, Maudlin makes three principal argumentative claims:

1. Topology faces substantial conceptual flaws as a foundation for the "submetrical" geometry of physical space.

2. The theory of linear structures is a novel alternative and genuine competitor to topology as a theory of this "submetrical" structure.

3. Moreover, the theory of linear structures is a superior alternative to topology because it is based on a sound analysis of our pre-theoretic concepts of "submetrical" geometry, such as "continuity," and thus should *replace* it (112).

Maudlin stresses that his complaints about standard topology are not about its mathematical soundness (2) or fruitfulness (52). Indeed, he thinks that standard topology can sometimes be fine for "metaphorical" spaces, such as a space of solutions to an equation, but not for geometrical ones, such as the Euclidean plane, where the relations between the points are about *closeness* rather than similarity. Maudlin declines to explicate this further, demurring that "Attempting to give explicit definitions is a dangerous business" (7), although he chides mathematicians for not recognizing the difference: "Mathematical practice has systematically ignored, and positively disguised, that distinction [between metaphorical and geometric spaces] for some time" (8).

Yet the book contains almost no discussion of the actual views of topologists, contemporary or past. This lack of connection with the vast literature on topology is a perplexing omission, for a reader unfamiliar with the history of the subject might get the impression that the conceptual problems Maudlin raises have gone unrecognized and unrectified in the mathematics community.[5]

[5] There an extremely brief (and therefore, unfortunately, superficial) discussion of diffeological and Frölicher spaces, as well as sheafs and topoi, in which Maudlin dismisses these approaches because they characterize the geometry of a space by mappings into it from another, well-understood geometrical space, instead of doing so "intrinsically". Now, the lines of a Linear Structure could be understood as mappings from ordered spaces, too, but Maudlin effec-

Many mathematicians since the beginnings of modern topology have in fact worked on a huge range of alternatives, often in response to exactly the kinds of concerns that Maudlin raises. To take one example, the *d-spaces* of Grandis (2009, 51), who in his recent monograph synthesizes his work on this class of mathematical objects (and related classes) since introducing them in 2002, are defined each as a set X equipped with a set d of distinguished paths, maps $a : [0, 1] \to X$, which include every constant map, are closed under concatenation, and are invariant under monotone transformations. The Theory of Linear Structures has begun from a similar starting point as that of d-spaces, and indeed most of the differences are cosmetic.[6]

Maudlin does not mention the work of Grandis in his monograph. Indeed, the only passages from topologists that Maudlin discusses come from two textbooks for undergraduates (Crossley, 2005; Gamelin and Greene, 1983/1999) and Wikipedia.[7] Maudlin never explains why he takes Wikipedia—"the gold standard for common opinion" (175)—and elementary textbooks to be the best source of conceptual clarity for the foundations of topology.[8] But one can perhaps fathom the reasons he might have when one examines his methodology. In doing so, it becomes clear that if Maudlin's book is audacious or outrageous, it is not quite for the reasons he imagines. It is not because, as he alleges, "the standard mathematical tools for analyzing physical geometry have, for over a hundred years, been the wrong tools," (2) but because he insists on certain views about the nature of and proper methodology for mathematics that mathematicians have since 1850 progressively all but abandoned.

While Maudlin does not quite devote a section of his book to articulating and defending his methodology, he nevertheless presents it clearly. His goal is an explication of what he takes to be *the* informal, pre-existing geometrical concepts, such as *the line*, *continuity*, *connectedness*, and even a region's being *open* or *closed*. His central standard of evaluation for this explication and of the adequacy of standard topology is the extent to which they meet "inviolable intuitions" (305), appeals to which appear on most pages of the book. Moreover, he assumes, apparently without comment, that intuition is univocal—that its objects are mind-independent, if imprecise, concepts, or *essences* of those concepts. (Two of the book's sections are even titled "The Essence of the Line" and "The Essence of Continuity.") Only once in the book does he seem, at first, to consider that his readers' intuitions might differ from his own: he bemoans that his criticisms of standard topology may not convince those "already familiar with the standard definitions [of topology, who] will have so internalized [them] that *those definitions express what they now mean by terms such as 'continuous'*" (3).[9] But, he continues, this is only because such readers have forgotten about

tively characterizes them in terms of the images of such mappings. All the above approaches can also dispense with mappings by describing only their images, so Maudlin's dismissal seems too quick.

[6]The main substantive difference between d-spaces and Linear Structures is that the distinguished paths of the former must be modeled on a dense, complete linear order—that is, one order isomorphic to a segment of the real line—while the lines of the latter can be dense but incomplete (i.e., order isomorphic to the rational line) or even discrete (i.e., order isomorphic to the integers).

[7]There are forty references cited in the bibliography, seven of which concern topology. Four of these are elementary textbooks on general topology, one is the foundational text in German by Hausdorff (1914), one is a standard reference in general topology, and one is Wikipedia.

[8]He does assert that the passages he quotes, which invoke phrases like "wiggling" a point to see that it has some "breathing" room, "are not mere window-dressing or inessential visual aids" (31) because the concepts of topology are built from these intuitive thought experiments. But this is to assume, dubiously, that the authors he quotes share his methodology (cf. p. 5) of building a mathematical theory on intuitions developed from simple examples.

[9]Original italics.

"the original, somewhat amorphous, concepts that stood in need of clarification."

Accordingly, at times Maudlin seems to imply that his whole Theory of Linear Structures is contained in the marriage of basic intuitions and his two central claims stated at the beginning of this review: "Everything flows from them in such a straightforward way that I am almost convinced that the reader could stop reading forthwith and, with sufficient patience and diligence, reconstruct most of what follows from these two propositions" (1). Intuition can lead one to accept an example or definition when it presents as expected, but it can also lead one to reject a possible definition when it leads to surprising examples (126).

But in other passages he remarks that intuition does not always impose a clear verdict on a particular example or choice of definition. In these cases, but only these cases—such as those of closed sets (136), the hierarchy of substructures (218), the tangent to a line (244), and the Euclidean plane (250)—there is some freedom in developing a theory along pluralist lines. The content of this three-valued ruling of intuition—accept, reject, or stay neutral—arises for Maudlin from specific, canonical, simple examples, and from the practice of drawing (121). Being *prosaic* examples and practices, they do not lift one's intuitive judgments to any great confidence, positive or negative, on many matters of precise theory development. So in these circumstances, when intuition is weak or mute, one should not settle on *the* explication of a concept.[10] But for Maudlin, especially in the beginning of the development of his theory, these circumstances are rare.

This perhaps explains why he looks solely to Wikipedia and elementary textbooks as reliable sources for the conceptual basis of standard topology: if one assumes that a viable mathematical theory must build on pre-theoretic intuitions that we all share, on the basis of our common stock of simple examples and practices of drawing, then Wikipedia and elementary textbooks arguably record and develop these intuitions. And if one assumes that these are univocal intuitions common to all, then to explore a variety of sources, including research articles and monographs that do not seem to invoke them—even indirectly—would be superfluous.

In building from simple examples and practices, Maudlin takes himself to be following an Aristotelian methodology for physical geometry: "Aristotle characterized the method of science as starting from those things that are clearer and more knowable to us and proceeding to those things that are clearer and more knowable in themselves. The mathematical elucidation of geometry must proceed in the same way" (3). By beginning from the sources of clear intuition—simple examples and drawing—Maudlin expects his theory to elucidate not just geometric concepts themselves, but the very structure of space and time. One of the eventual aims of his project is the development of "primitive mathematical properties that seem like plausible candidates for properties that physical items could actually have. There is a longstanding puzzle about why mathematics should provide such a powerful language for describing the physical world. The most satisfying answer to such a question is: *Because the physical world literally has a mathematical structure*" (52).[11] He continues that this allows for the possibility "that physical spacetime *literally* has a geometrical structure," meaning that "the physical world can literally be a model of the geometric axioms, and hence literally be a geometric object."

Accordingly, with the ancients, Maudlin explicitly wants to "make a forceful separation be-

[10]For instance, he writes that "our intuitive sense has not had to decide between these distinct properties as the 'essence' of closedness. Which of these various definitions of a closed set should we accept? There is no substantive issue here. The different concepts are all well-defined, and which is the most appropriate to invoke at any moments [*sic*] depends on the particular task at hand." (136)

[11]Original italics.

tween the arithmetical and geometrical objects" (9), rejecting the modern synthesis of geometric and algebraic ideas found in the arithmetic continuum of Dedekind (1901/1963) and in Weierstrass' ϵ-δ definition of the continuity of a function, the abstraction of which forms the definition of continuity for functions between general topological spaces. In their work, Dedekind and Weierstass wished to find a firmer but also more fruitful foundation for analysis in purely arithmetic notions. But Maudlin believes this was a mistake: "For several millenia, the actual space we live in was believed to be a three-dimensional Euclidean space, but no one ever imagined that the space we live in consists in ordered triples of real numbers. Such a proposal makes no sense whatever" (8). He admits that ordered triples of real numbers can, of course, be used to represent Euclidean space *metaphorically*, but this is ultimately unsatisfactory because it "imposes a screen of mathematical representation between us and the object in which we are interested" (9), that is, it cannot provide a literal description of Euclidean space. Through this lens of mathematics as metaphysics, he believes that recent mathematics confuses, for example, the Euclidean plane with $\mathbb{R}^2$, as the latter has an origin but the former does not (8), and derides the use of equivalence classes of coordinate charts in defining differentiable manifolds as so much obfuscating superfluousness (217).

Now, as much as Maudlin's Theory of Linear Structure successfully avoids these issues, on these examples Maudlin is simply mistaken: mathematicians (and mathematical physicists) are careful to distinguish $\mathbb{R}^2$ as a field, as a partially ordered space, as a manifold, as an affine space, etc. Moreover, modern presentations of differential geometry needn't have recourse to the algebraic structure of real numbers, for they may use maximal atlases of charts, or pseudogroups of transition functions, or structure sheaves, etc. But Maudlin's methodological commitment is clear: a properly precise and accurate mathematics of physical geometry must cohere with intuition and eschew any numerical or arithmetic concepts so that it can be a literal description of real, ontologically robust physical space.

This position on the relationship between mathematics and the world was more at home during the height of the early modern period than today. From the brink of the scientific revolution in the sixteenth century through its height and into the eighteenth, mathematics and the sciences of motion were not distinguished. Indeed, Maudlin cites approvingly both Galileo's famous line that nature's book is written in the language of mathematics (vi, 52) and Newton's criticisms of Descartes' arithmetic methods for geometry as confounding two subjects (16). For late-seventeenth- and eighteenth-century mathematicians, too, intuition was an accepted aspect of methodology, but primarily because of the enormous mathematical progress it had afforded and continued to promise. The successful application of this mathematics vouchsafed its truth because "mathematics was simply unearthing the mathematical design of the universe" (Kline, 1972, 619). Indeed, "Descartes, Newton, Euler, and many others believed mathematics to be the accurate description of real phenomena . . . they regarded their work as the uncovering of the mathematical design of the universe" (Kline, 1972, 1028). But the nineteenth century increasingly saw mathematicians grappling with mathematics that had no such apparent connection: spaces of dimension greater than three, negative and complex numbers, and non-Euclidean geometries. These developments rapidly changed most mathematicians' conceptions of their discipline, so that by the mid-nineteenth century the idea of uncovering the mathematical script of Nature's book was seriously contested. By century's end, it was gone, replaced by a newly pure mathematics of "freestanding abstract models that [in application to physical phenomena] resemble the world in ways that are complex and sometimes not fully understood" (Maddy, 2008, 33).[12] As a part of the vanguard of this movement into the twentieth century, topology was central to resolving "the conflict between geometry and analysis by obscuring the boundaries between the two disciplines. One of the fundamental measures of progress in mathematics is unification. Point set topology represents a major achievement in this direction" (Manheim, 1964, 143). Thus topology, as a mathematical discipline, developed historically as part of the broader reaction against the kind of methodology Maudlin now espouses.

This is why I wrote that if Maudlin's book is audacious or outrageous, it is not quite for the reasons he imagines. It is not because he is calling for new and perhaps substantially different mathematics in the service of a better model for geometry. In that project he would be joining many distinguished mathematicians. Nor is it because he invokes intuition in his motivations. There, too, he would not meet universally with quarrel: the extremely influential mid-twentieth-century mathematician Alexander Grothendieck too stressed the importance of the "topological intuition of shapes" (Schneps and Lochak, 1997, 262) for revealing "the necessity of new foundations for 'geometric' topology" (Schneps and Lochak, 1997, 264) in his famous 1984 proposal, *Equisse d'un Programme*. It is rather, for Maudlin, the evidential centrality of intuition, that intuition's assumed unique mathematical object, and the goal of geometry of being a literal metaphysics of space that set his project outside the mathematical mainstream.

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[12]For more on these developments and others leading to the tectonic shift in mathematics in the nineteenth century, see Kline (1972), Maddy (2008), and Gray (2014), from whom I have drawn the above account.

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