Light Clocks and the Clock Hypothesis

Samuel C. Fletcher

Foundations of Physics
An International Journal Devoted to
the Conceptual Bases and Fundamental
Theories of Modern Physics

ISSN 0015-9018
Volume 43
Number 11

Found Phys (2013) 43:1369-1383
DOI 10.1007/s10701-013-9751-3

Springer

Found Phys (2013) 43:1369–1383
DOI 10.1007/s10701-013-9751-3

Light Clocks and the Clock Hypothesis

Samuel C. Fletcher

Received: 15 February 2013 / Accepted: 27 September 2013 / Published online: 12 October 2013

Abstract The clock hypothesis of relativity theory equates the proper time experienced by a point particle along a timelike curve with the length of that curve as determined by the metric. Is it possible to prove that particular types of clocks satisfy the clock hypothesis, thus genuinely measure proper time, at least approximately? Because most real clocks would be enormously complicated to study in this connection, focusing attention on an idealized light clock is attractive. The present paper extends and generalized partial results along these lines with a theorem showing that, for any timelike curve in any spacetime, there is a light clock that measures the curve's length as accurately and regularly as one wishes.

Keywords General relativity · Clock hypothesis · Light clock · Born rigid · World-function

1 Introduction

The clock hypothesis (CH) of relativity theory equates the proper time experienced by a point particle along a timelike curve with the length of that curve as determined by the metric. More formally,[1] given a Lorentzian manifold (M, g_{ab}) with metric signature $(+ - \cdots -)$ and an interval $I \subseteq \mathbb{R}$, if $\gamma : I \to M$ is a timelike curve with tangent field ξ^a, then its associated proper time τ_γ (relative to g_{ab}) is just

$$\tau_\gamma = \|\gamma\| = \int_I \sqrt{g_{ab}\xi^a\xi^b}\, ds. \tag{1}$$

[1] Throughout I use geometrical units ($c = 1$) and the "abstract index" notation: roman super- and subscripts indicate where vector and tensor fields reside. (See, for example, Sect. 1.4 of Malament [1].)

S.C. Fletcher (✉)
Department of Logic and Philosophy of Science, University of California, Irvine,
3151 Social Science Plaza A, Irvine, CA 92697-5100, USA
e-mail: scfletch@uci.edu

While the word "clock" may evoke specific human inventions for timekeeping, the CH is in fact completely general: any physical process, insofar as it can be idealized as pointlike and taking place along a worldline, can be considered as a kind of clock to the extent that the state of the process is systematically correlated with the dynamical time elapsed.

The CH has been verified at high ($\sim$0.1 %) precision for the decay rates of muons undergoing centripetal accelerations of $\sim 10^{18}$ g [2]. Nevertheless it has not gone unquestioned in a wider context of experimental conditions (e.g., see [3, 4]), and one might inquire under what such conditions Eq. (1) holds true. Is it possible to prove that particular types of clocks satisfy the clock hypothesis, thus genuinely measure proper time, at least approximately?[2]

Such an analysis for most real clocks is, and must be, enormously complicated. Thus some previous works analyzing, e.g., a harmonic oscillator [7, 8], a pendulum [9, pp. 394–395], or a muon (in Minkowski spacetime) [10], have required the use of extensive approximations regarding the dynamics, making the study of an idealized *light clock* attractive. Briefly, the simplest form of a light clock consists of a light ray bouncing between two parallel perfectly reflective mirrors separated by a distance d. If the light ray completes n round-trips between the mirrors, then the clock has recorded an elapsed time of $2nd$. Importantly, decreasing the separation between the mirrors increases the clock's resolution. Because one can represent the mirrors using timelike curves and the bouncing light ray as a set of null geodesics, one can represent a light clock's dynamics in a relativistic spacetime without appealing to Einstein's field equations. (One might say that the description becomes purely kinematical.)

Maudlin [11, Ch. 5] has recently argued that, given some additional assumptions, one can prove that the quantity an inertially moving light clock measures in Minkowski spacetime is the proper time (Eq. (1)) along its worldline. He supposes that the mirrors of the light clock are connected by a body that has an "equilibrium state," a configuration with particular proper distances between its parts to which the body tends to return if gently perturbed. Because there is no geodesic deviation in Minkowski spacetime, such equilibrium states correspond to particular trajectories along a timelike geodesic congruence. Without a detailed account of the constitution of the body, one cannot say much about how the clock behaves while accelerated, but if these bouts of acceleration are sufficiently small in magnitude, the body will return to its equilibrium state. And for light clocks in such states, it is elementary to prove that they measure proper time.

As remarkable as this argument is, its scope is somewhat restricted. The key step of using equilibrium states depends on special symmetries of Minkowski spacetime, but the CH plays just as an important role in the general theory of relativity as it does in the special. Moreover, what "state" an extended body is in will in general depend on the choice of reference frame. One might therefore require Born rigidity, saying that

[2] I should emphasize that the present work is to be distinguished from that of Marzke and Wheeler [5, 6] and others following them. In those works, the authors investigate the related question of how, *given* that the CH holds, one can construct a clock from minimal elements. By contrast, here I am concerned instead with showing in detail *that* the CH holds for a particular construction.

a light clock is in equilibrium in some spacetime region just when all inertial frames instantaneously co-moving with each material part of the clock agree that it is. Such states require the light clock to undergo sufficiently long stretches of inertial motion. But one would also like to say more about the behavior of light clocks undergoing acceleration, since it is reasonable to assume that most of the time, most parts of most clocks are undergoing *some* acceleration.

Treating the effect of acceleration on a light clock has already been studied in connection with the CH, but so far in a somewhat restricted manner. Marder [12, pp. 91–93] gives a heuristic analysis for a light clock in Minkowski spacetime accelerated along the perpendicular to its mirrors (i.e., parallel to the light ray), concluding that it will be ideal if the distance between the mirrors is sufficiently small. Kowalski [13], reaching much the same conclusions, extended this analysis quantitatively, as well as to uniform circular motion and linear motion perpendicular to the direction of the light ray, including an error analysis of the effects of acceleration up to terms proportional to c^{-3}. Only the much earlier work of Gautreau and Anderson [14, 15] has considered the arbitrary motion of light clocks in general (non-flat) spacetimes, although they focused on giving an existence proof for ideal light clocks instead of an error analysis. They do verify that light clocks measure proper time, but only under the additional assumptions that the clock is Born rigid and that the acceleration is negligible.

The present paper generalizes these latter results, indicating a direction in which one can extend Maudlin's argument to light clocks undergoing arbitrary acceleration in arbitrary spacetimes. The central result is a theorem showing that, given any $C^{(2)}$ timelike curve in any spacetime, there is a sufficiently small light clock that measures the proper time of that curve as accurately and as regularly as one wishes. In other words, the error in the total proper time measured for any interval, and the maximum difference in proper time measured between any two pairs of consecutive light bounces, can be made as small as desired. This improves upon the calculations of Gautreau and Anderson [14, 15] by assuming neither that the clock is Born rigid nor that its acceleration is negligible. Like Gautreau and Anderson (and Maudlin), I have focused on providing an existence proof for sufficiently ideal light clocks rather than an error analysis for any particular light clock. I will postpone further discussion of the theorem's interpretation to Sect. 3.

2 The Central Theorem

In what follows, M will always designate some (point set of a) Lorentzian manifold. The proof of the central theorem turns on a lemma that uses techniques developed extensively by Synge [16], in particular, certain analytic features of the *world-function*, whose rigorous definition requires some work to state. Much of this definitional work is based on pp. 5, 126–127 of Sachs and Wu [17].

Definition 1 The *exponential map* at $p \in M$, $\exp_p : U_p \to M$, is defined on a subset U_p of the tangent space T_pM as follows. First, $\mathbf{0} \in U_p$ and $\exp_p \mathbf{0} = p$. Then any nonzero $\alpha^a \in U_p$ if and only if there is a geodesic $\gamma : [0, 1] \to M$ with tangent vector

α^a at p such that $\gamma(0) = p$. Finally, for such nonzero $\alpha^a \in U_p$, $\exp_p \alpha^a = \gamma(1)$, which is well-defined since the geodesic γ corresponding to α^a is unique.

Informally, the exponential map at a point maps a tangent vector to a point in the manifold connected to the original point by a geodesic; formally, a curve $\gamma(t) = \exp_p(t\mu^a)$ that can be reparameterized so as to be a geodesic is generated through p for every nonzero $\mu^a \in U_p$.

Definition 2 Given $p \in M$, an open neighborhood $U_0 \subseteq T_p M$ containing the zero vector is called *normal* if and only if

1. $(\exp_p)_{|U_0}$ is well-defined and a diffeomorphism onto its image, and
2. U_0 is closed under scalar multiplication by $t \in [0, 1]$.

Definition 3 A *simply convex* neighborhood U on M is a nonempty open set $U \subseteq M$ such that, if $p \in U$, then $U = \exp_p U_0$ for some normal neighborhood $U_0 \subseteq T_p M$.

Every $p \in M$ has a simply convex neighborhood. It follows that, for distinct p, $q \in U$, where U is a simply convex neighborhood in M, there is a unique geodesic $\gamma : [0, 1] \to U$ connecting points p and q. (See, respectively, Prop. 5.7 on p. 130 and Prop. 3.31 on p. 72 of O'Neill [18].) This uniqueness allows for the world-function to be well-defined:

Definition 4 The *world-function* on a simply convex neighborhood $U \subseteq M$ is a map $\Omega(p, q) : U \times U \to \mathbb{R}$, defined by $\Omega(p, q) = \frac{1}{2}(\exp_q^{-1}(p))^a(\exp_q^{-1}(p))_a$.[3]

One can show that:[4]

1. $\Omega(p, q)$ is symmetric, and smooth in both of its arguments.
2. If $p \neq q$, then $\Omega(p, q) = \frac{1}{2}\bar{\xi}^a\bar{\xi}_a\|\bar{\gamma}_{pq}\|^2$, where $\bar{\gamma}_{pq}$ is the geodesic with unit tangent vector $\bar{\xi}^a$ joining p and q.[5] (That is, $q = \exp_p(\|\bar{\gamma}_{pq}\|\bar{\xi}^a)$.) Otherwise $\Omega(p, q) = \Omega(p, p) = 0$.
3. If one fixes $p \in M$ and adopts the notation $\overset{p}{\Omega}(q) \equiv \Omega(p, q)$, then clearly $\overset{p}{\Omega}(q)$ is a smooth scalar field with first covariant derivative

$$(\nabla_a \overset{p}{\Omega})(q) = \big(\bar{\xi}^b\bar{\xi}_b\|\bar{\gamma}_{pq}\|\bar{\xi}_a\big)_{|q}. \tag{2}$$

Given two points in its domain, the world-function returns half the signed squared magnitude of the (unique) geodesic connecting those points, the sign indicating

[3] Sachs and Wu [17] define a function $\Phi(p, q) = 2\Omega(p, q)$, calling it the *geometric energy function*, despite admitting that using the term "energy" here is somewhat misleading. Needless to say, the multiplicative factor does not materially change the results at hand.

[4] Again, see pp. 126–127 of Sachs and Wu [17].

[5] If p and q are null-related, there is no unit $\bar{\xi}^a$, but then $\Omega(p, q) = 0$ independently of the choice of tangent vector.

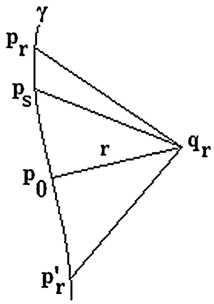

Fig. 1 Illustration of the lemma. The point q_r is connected to p_0, which is on (the image of) a timelike *curve* γ, by a geodesic of length r orthogonal to γ. Points p_r and p'_r on (the image of) γ are connected to q_r by null geodesics, while point p_s ranges in between

whether the geodesic is timelike or spacelike. (The particular sign, $+$ or $-$, depends upon whether one adopts a metric of signature $(+-\cdots-)$ or $(-+\cdots+)$, respectively, but in either case the world function vanishes for null geodesics.) To get a sense for the meaning of Eq. (2), consider the case where (M, g_{ab}) is Minkowski spacetime, and χ^a is the position field relative to $p \in M$. Then the world-function can be written $\overset{p}{\Omega}(q) = \frac{1}{2}(\chi^a \chi_a)_{|q}$ and its covariant derivative reduces to $(\nabla_a \overset{p}{\Omega})(q) = (\chi_a)_{|q}$.

In what follows, we will also need an expression for the second covariant derivative of the world-function. Given $p \in U$, for all timelike-related $q \in U$, one can calculate from Eq. (2) that

$$\begin{aligned}(\nabla_b \nabla_a \overset{p}{\Omega})(q) &= \big[\nabla_b\big(\bar{\xi}^c \bar{\xi}_c\big)\|\bar{\gamma}_{pq}\|\bar{\xi}_a + \big(\bar{\xi}^c \bar{\xi}_c\big)\big(\nabla_b\|\bar{\gamma}_{pq}\|\big)\bar{\xi}_a + \bar{\xi}^c \bar{\xi}_c \|\bar{\gamma}_{pq}\| \nabla_b \bar{\xi}_a\big]_{|q} \\ &= \big(\bar{\xi}^c \bar{\xi}_c\big)\big(\bar{\xi}_b \bar{\xi}_a + \|\bar{\gamma}_{pq}\| \nabla_b \bar{\xi}_a\big)_{|q}, \end{aligned} \tag{3}$$

since $\nabla_b(\bar{\xi}^c \bar{\xi}_c) = \mathbf{0}$ and $\nabla_b \|\bar{\gamma}_{pq}\| = \bar{\xi}_b$. The calculation for spacelike-related p is exactly the same, hence the smoothness of the world-function must extend Eq. (3) to null-related q, so that it holds in fact for *all* $q \in U$.

We are now ready to state the aforementioned lemma, which one can understand as proving the limiting result desired, but restricted to a single bounce of the light ray. (For illustration, see Fig. 1.)

Lemma 1 *Let $\gamma : I \to M$ be a $C^{(2)}$ timelike curve, parameterized by arc length, with tangent vector field ξ^a. Suppose $\gamma(s_0) = p_0$ and $\rho^a \in T_{p_0}M$ is a spacelike unit vector orthogonal to ξ^a that determines a geodesic $\exp_{p_0}(r\rho^a) = q_r$ parameterized by arc length. For sufficiently small $r > 0$, let $\gamma(s_r) = p_r$ and $\gamma(s'_r) = p'_r$ be the unique distinct points connected to q_r by null geodesics, with $s_r > s'_r$. (That is, $p_r = \exp_{q_r}(t\mu^a)$ and $p'_r = \exp_{q_r}(u\nu^a)$ for some $t, u \neq 0$ and nonzero null $\mu^a, \nu^a \in T_{q_r}M$.) Then if s_0 is restricted to any compact set $S \subset I$, $(s_r - s'_r)/r \to 2$ uniformly on S as $r \to 0$.*

Proof Let $p_s = \gamma(s)$ for $s_0 < s < s_r$. Since the world-function $\overset{q_r}{\Omega}(p_s)$, defined on some simply convex neighborhood of p_0, is a smooth scalar function on the real interval $[s_0, s_r]$, we can expand it to second order about s_0. Suppressing the argument of the covariant derivatives of $\overset{q_r}{\Omega}(p_s)$, we have that

$$\overset{q_r}{\Omega}(p_s) = \overset{q_r}{\Omega}(p_0) + (s - s_0)\big(\xi^a \nabla_a \overset{q_r}{\Omega}\big)_{|p_0} + \frac{1}{2}(s - s_0)^2\big[\xi^b \nabla_b\big(\xi^a \nabla_a \overset{q_r}{\Omega}\big)\big]_{|p_0} + O\big[(s - s_0)^3\big].$$

Now $[\xi^b \nabla_b(\xi^a \nabla_a \overset{q_r}{\Omega})]_{|p_0} = [(\xi^b \nabla_b \xi^a)\nabla_a \overset{q_r}{\Omega} + \xi^b \xi^a \nabla_b \nabla_a \overset{q_r}{\Omega}]_{|p_0}$, while Eq. (2) yields $(\nabla_a \overset{q_r}{\Omega})_{|p_0} = (-r\rho_a)_{|p_0}$, so this becomes

$$\begin{aligned}\overset{q_r}{\Omega}(p_s) &= \overset{q_r}{\Omega}(p_0) - (s - s_0)\big(r\xi^a \rho_a\big)_{|p_0} \\ &\quad + \frac{1}{2}(s - s_0)^2\big(-r\rho_a \xi^b \nabla_b \xi^a + \xi^b \xi^a \nabla_b \nabla_a \overset{q_r}{\Omega}\big)_{|p_0} + O\big[(s - s_0)^3\big] \\ &= -\frac{1}{2}r^2 - \frac{1}{2}(s - s_0)^2\big(r\rho_a \xi^b \nabla_b \xi^a\big)_{|p_0} \\ &\quad + \frac{1}{2}(s - s_0)^2\big(\xi^b \xi^a \nabla_b \nabla_a \overset{q_r}{\Omega}\big)_{|p_0} + O\big[(s - s_0)^3\big], \end{aligned} \tag{4}$$

as $\overset{q_r}{\Omega}(p_0) = -\frac{1}{2}r^2$ and $(\xi^a \rho_a)_{|p_0} = 0$. Focusing on the term with a second-order derivative, we have

$$\big(\xi^b \xi^a \nabla_b \nabla_a \overset{q_r}{\Omega}\big)_{|p_0} = \big(\xi^b \xi^a \nabla_b \nabla_a \overset{q_r}{\Omega}\big)(p_0) = \big(\xi^b \xi^a \nabla_b \nabla_a \overset{q_r}{\Omega}\big)(p_0)_{|r=0} + O(r), \tag{5}$$

using the fact that $\overset{q_r}{\Omega}(p_0)$ is a smooth function of r to expand it to zeroth order about $r = 0$. Note that $q_0 = p_0$, so due to the smoothness of the world-function, one can calculate the value of the zeroth order term by taking a limit along an arbitrary continuous curve ending at p_0. In particular, let $\tilde{\xi}^a$ be any continuous extension of ξ^a to a neighborhood of p_0 so that the curve $\hat{\gamma}(t) = \exp_{p_0}(t\xi^a) = \hat{p}_t$ is a timelike geodesic parameterized by arc length with tangent vector $\hat{\xi}^a$. Then, invoking Eq. (3),

$$\begin{aligned}\big(\xi^b \xi^a \nabla_b \nabla_a \overset{p_0}{\Omega}\big)(p_0) &= \lim_{t\to 0}\big(\tilde{\xi}^b \tilde{\xi}^a \nabla_b \nabla_a \overset{p_0}{\Omega}\big)(\hat{p}_t) \\ &= \lim_{t\to 0}\big[\tilde{\xi}^b \tilde{\xi}^a \big(\hat{\xi}^c \hat{\xi}_c\big)\big(\hat{\xi}_b \hat{\xi}_a + \|\hat{\gamma}_{p_0 \hat{p}_t}\| \nabla_b \hat{\xi}_a\big)\big]_{|\hat{p}_t} \\ &= \lim_{t\to 0}\big[\big(\tilde{\xi}^a \hat{\xi}_a\big)^2 + t\tilde{\xi}^a \tilde{\xi}^b \nabla_b \hat{\xi}_a\big]_{|\hat{p}_t}.\end{aligned}$$

Since the geodesic equation requires that $(\tilde{\xi}^b \nabla_b \hat{\xi}_a)_{|\hat{p}_t} \to \mathbf{0}$ as $t \to 0$, we have that

$$\big(\xi^b \xi^a \nabla_b \nabla_a \overset{p_0}{\Omega}\big)(p_0) = \lim_{t\to 0}\big(\tilde{\xi}^a \hat{\xi}_a\big)^2_{|\hat{p}_t} = \big(\xi^a \xi_a\big)^2_{|p_0} = 1.$$

Thus we can write Eq. (4) as

$$\overset{q_r}{\Omega}(p_s) = -\frac{1}{2}r^2 - \frac{1}{2}(s-s_0)^2\big(r\rho_a\xi^b\nabla_b\xi^a\big)_{|p_0} + \frac{1}{2}(s-s_0)^2\big(1+O(r)\big) + O\big[(s-s_0)^3\big].$$

Now taking the limit $s \to s_r$ gives

$$0 = -\frac{1}{2}r^2 - \frac{1}{2}(s_r-s_0)^2\big(r\rho_a\xi^b\nabla_b\xi^a\big)_{|p_0} + \frac{1}{2}(s_r-s_0)^2\big(1+O(r)\big) + O\big[(s_r-s_0)^3\big],$$

which we can rearrange to yield

$$\frac{r^2}{(s_r-s_0)^2} + O(s_r-s_0) = 1 + O(r), \tag{6}$$

after absorbing the term containing $r\rho_a\xi^b\nabla_b\xi^a$ into $O(r)$. The supremum over $s_0 \in S$ of the left-hand side must exist for compact S, as it is a continuous function of s_0, and be equal to some function of the form $1+O(r)$. But such a function clearly converges to 1 in the $r \to 0$ limit, so for every $\epsilon > 0$, there is some r_0 such that, for $0 < r < r_0$,

$$\sup_{s_0\in S}\left|\frac{r^2}{(s_r-s_0)^2} + O(s_r-s_0) - 1\right| < \epsilon,$$

which holds just in case $r^2/(s_r-s_0)^2 + O(s_r-s_0) \to 1$ uniformly on S as $r \to 0$. Noting that $s_r \to s_0$ as $r \to 0$, we thus have that $\lim_{r\to 0} r^2/(s_r-s_0)^2 = 1$ uniformly on S, or equivalently, $\lim_{r\to 0}(s_r-s_0)/r = 1$ uniformly on S, since by definition $r > 0$ and $s_r > s_0$. A similar argument shows that $\lim_{r\to 0}(s_0-s'_r)/r = 1$ uniformly on S, hence

$$\lim_{r\to 0}\frac{s_r-s'_r}{r} = \lim_{r\to 0}\frac{s_r-s_0}{r} + \lim_{r\to 0}\frac{s_0-s'_r}{r} = 2$$

uniformly on S. □

Since the foregoing lemma concerns the behavior of a single "bounce" or tick of a light clock in an arbitrary spacetime, one might wonder where the Riemann curvature $R^a{}_{bcd}$ shows up in the analysis. The answer is that it arises in terms proportional to r^2 in the expansion given by Eq. (5). In particular, one can show [16, pp. 125–126] that this expansion can be written more fully as

$$\big(\xi^b\xi^a\nabla_b\nabla_a\overset{q_r}{\Omega}\big)(p_0) = \big(\xi^b\xi^a\nabla_b\nabla_a\overset{q_r}{\Omega}\big)(p_0)_{|r=0} + r\big(\rho^c\nabla_c\big(\xi^b\xi^a\nabla_b\nabla_a\overset{q_r}{\Omega}\big)\big)(p_0)_{|r=0} + \frac{1}{2}r^2\big(\rho^d\nabla_d\big(\rho^c\nabla_c\big(\xi^b\xi^a\nabla_b\nabla_a\overset{q_r}{\Omega}\big)\big)\big)(p_0)_{|r=0} + O\big(r^3\big)$$

$$= 1 + \frac{1}{3} r^2 \big(R_{abcd} \xi^a \rho^b \xi^c \rho^d \big)_{|p_0} + O\big(r^3\big).$$

One might say roughly that this curvature term describes deviations from ideal behavior that arise from tidal effects on the shape of the light clock.[6] Of course, for the present purposes of proving a limiting result rather than analyzing a particular light clock, it suffices to observe that these (and higher order) effects of curvature do become negligible for sufficiently small r.

Now, the statement of the theorem is more concise when we introduce the following novel terminology.

Definition 5 Let $\gamma : I \to M$ be a $C^{(2)}$ timelike curve and suppose $I' \subseteq I$ is a closed interval. Then we say that (the set of images of) a one-parameter family of curves $\{\overset{\alpha}{\gamma}\}_{\alpha \in \mathbb{N}}$, is a *convergent companion family* to $\gamma[I']$ when:

1. For each α, one can define a (not necessarily smooth) unit spacelike vector field $\overset{\alpha}{\rho}{}^a$ on $\gamma[I']$, orthogonal to (the tangent vector of) γ, called the family's *orientation vector field*, and a continuous non-zero scalar field $\overset{\alpha}{r}$ on $\gamma[I']$, called the *scalar radius field*, such that, for each $p \in \gamma[I']$, $\exp_p(\overset{\alpha}{r}\overset{\alpha}{\rho}{}^a) = \overset{\alpha}{\gamma}(t_p)$ for some unique t_p in the domain of $\overset{\alpha}{\gamma}$.
2. Additionally, $\lim_{\alpha \to \infty} \overset{\alpha}{r} = 0$, the zero scalar field.

For generality, there is no requirement on the smoothness of the curves $\overset{\alpha}{\gamma}$, although for physical examples one might require more than this. The first condition specifies the sense in which $\{\overset{\alpha}{\gamma}\}_{\alpha \in \mathbb{N}}$ is a *companion family*: each point of $\gamma[I']$ has a unique "companion" on (the image of) each $\overset{\alpha}{\gamma}$ that is completely specified by the fields $\overset{\alpha}{\rho}{}^a$ and $\overset{\alpha}{r}$. While it may seem that demanding $\overset{\alpha}{r}$ to be nonzero is a merely technical requirement, it has a natural interpretation as it is used in the theorem. There, the curves of the companion family will represent the spacetime locations of one of the "mirrors" in a collection of light clocks recording elapsed time along the "mirror" $\gamma[I']$. Requiring $\overset{\alpha}{r}$ to be nonzero amounts to ensuring that there is a non-zero distance between the mirrors, hence that the "photon bouncing" is always well-defined. (It will also serve an important role in some of the following definitions.) The second condition specifies the sense in which the family is convergent. Requiring that $\alpha \in \mathbb{N}$ in the definition is conventional, since here α just serves as an index to label the curves in the convergent companion family. Thus replacing $\mathbb{N}$ with any other infinite index set would work equally well (provided one adjusts the $\alpha \to \infty$ limit accordingly, if necessary).

It is clear that, given some timelike curve $\gamma : I \to M$ and a closed interval $[a, b] \subseteq I$, there always exists some convergent companion family to $\gamma[a, b]$. For,

[6] If one were analyzing a concrete light clock where the spacelike unit vector $\rho^a \in T_{p_0}M$ pointing to the "mirror" is extended to the whole image of γ, and the tangent vector ξ^a is extended via Fermi–Walker transport to a neighborhood of the clock, then one can use the equation of geodesic deviation to write this curvature term as $\xi_a \rho^c \nabla_c (\rho^b \nabla_b \xi^a)$. However, one should take care not to interpret this as an "acceleration" in any strict sense, since the derivatives are along spacelike instead of timelike geodesics.

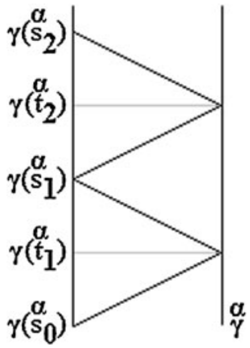

Fig. 2 Illustration of the definition of the bounce number $\overset{\alpha}{n}$ and proper time sequence $\overset{\alpha}{S}$ for a simple *curve* γ and a sample companion *curve* $\overset{\alpha}{\gamma}$. The *dark lines* between γ and $\overset{\alpha}{\gamma}$ indicate null geodesics and the *gray lines* spacelike geodesics orthogonal to $\overset{\alpha}{\gamma}$

supposing $\rho^a \in T_{\gamma(a)}M$ is some spacelike unit vector orthogonal to (the tangent vector of) γ at $\gamma(a)$, one can generate an orientation vector field, constant over α, by Fermi–Walker transport of ρ^a, and set a constant scalar radius field as $\overset{\alpha}{r} = 1/\alpha$ for sufficiently large α. In particular, if the α are large enough then the image of each curve of the resulting convergent companion family lies within the union of simply convex neighborhoods of the points of $\gamma[a,b]$ that are sufficiently small to guarantee that, for each $p, q \in \gamma[a,b]$, $\exp_p(\overset{\alpha}{r}\overset{\alpha}{\rho}{}^a) = \exp_q(\overset{\alpha}{r}\overset{\alpha}{\rho}{}^a)$ if and only if $p = q$.

Generally, when the (images of the) curves of a convergent companion family to $\gamma[a,b]$ lie within a such a union of simply convex neighborhoods, we may associate with each $\overset{\alpha}{\gamma}$ in the family a *bounce number* $\overset{\alpha}{n}$ and a *proper time sequence* $\overset{\alpha}{S}$, defined recursively as follows. (See Fig. 2 for illustration.)

Definition 6 For each $\alpha \in \mathbb{N}$, let $\overset{\alpha}{\gamma}$ be given.

1. For the base case, let $\overset{\alpha}{S}_0 = \{\overset{\alpha}{s}_0\}$, where $\overset{\alpha}{s}_0 = a$.
2. If there exist distinct points $\gamma(\overset{\alpha}{t}_k)$ and $\gamma(\overset{\alpha}{s}_k)$ such that $\gamma(\overset{\alpha}{s}_{k-1})$, $\exp_{\gamma(\overset{\alpha}{t}_k)}(\overset{\alpha}{r}\overset{\alpha}{\rho}{}^a)$, and $\gamma(\overset{\alpha}{s}_k)$ are sequentially connected by null geodesics, with each of $\overset{\alpha}{s}_{k-1} < \overset{\alpha}{t}_k < \overset{\alpha}{s}_k$ in $[a,b]$, then put $\overset{\alpha}{S}_k = \overset{\alpha}{S}_{k-1} \cup \{\overset{\alpha}{t}_k, \overset{\alpha}{s}_k\}$. (It follows from well-known results (see Prop. 5.2.3 of Sachs and Wu [17]) that, if $\overset{\alpha}{r}$ is sufficiently small, such points are guaranteed to exist.) Otherwise, put $\overset{\alpha}{S}_k = \overset{\alpha}{S}_{k-1}$.
3. Put $\overset{\alpha}{n} = \min\{k : \overset{\alpha}{S}_k = \overset{\alpha}{S}_{k+1}\}$ and $\overset{\alpha}{S} = \overset{\alpha}{S}_{\overset{\alpha}{n}}$.

Note that, by construction, the elements of any $\overset{\alpha}{S}$ constitute an ordered sequence: $\overset{\alpha}{s}_0 < \overset{\alpha}{t}_1 < \overset{\alpha}{s}_1 < \cdots < \overset{\alpha}{t}_{\overset{\alpha}{n}} < \overset{\alpha}{s}_{\overset{\alpha}{n}}$. Also, in order for the bounce numbers and proper time sequences of each companion curve to be well-defined in step 3, there must be some k such that $\overset{\alpha}{S}_k = \overset{\alpha}{S}_{k+1}$ for each α. To see why this is the case, suppose otherwise for some arbitrary α. Then we have an infinite strictly monotone increasing sequence $\overset{\alpha}{s}_0 < \overset{\alpha}{t}_1 < \overset{\alpha}{s}_1 < \cdots < \overset{\alpha}{t}_k < \overset{\alpha}{s}_k < \cdots$ that is also bounded by b, hence must converge as $k \to \infty$. But the Cauchy criterion then implies that $\lim_{k\to\infty} \overset{\alpha}{s}_k - \overset{\alpha}{t}_k = 0$. This difference bears a relationship to the scalar radius field $\overset{\alpha}{r}$ revealed by some calculations from the proof of the lemma. In particular, applying Eq. (6) shows that

$$\lim_{k\to\infty}\big(1+O\big[\overset{\alpha}{r}\big(\gamma(\overset{\alpha}{t}_k)\big)\big]\big)=\lim_{k\to\infty}\left(\left(\frac{\overset{\alpha}{r}(\gamma(\overset{\alpha}{t}_k))}{\overset{\alpha}{s}_k-\overset{\alpha}{t}_k}\right)^2+O[\overset{\alpha}{s}_k-\overset{\alpha}{t}_k]\right)$$
$$=\lim_{k\to\infty}\left(\frac{\overset{\alpha}{r}(\gamma(\overset{\alpha}{t}_k))}{\overset{\alpha}{s}_k-\overset{\alpha}{t}_k}\right)^2.$$

The left-hand side is clearly finite, but the denominator on the right-hand side vanishes as $k\to\infty$, so the numerator must as well. However, this is a contradiction, for by definition $\overset{\alpha}{r}(\gamma(s))$ is a strictly positive continuous function over the compact set $[a,b]$.

We are now ready to state the theorem.[7]

Theorem 1 *Let $\gamma: I\to M$ be a $C^{(2)}$ timelike curve parameterized by arc length, and let $I'\subset I$ be a closed interval. Consider $\{\overset{\alpha}{\gamma}\}_{\alpha\in\mathbb{N}}$, some arbitrary convergent companion family to $\gamma[I']$ with associated scalar radius fields $\overset{\alpha}{r}$, bounce numbers $\overset{\alpha}{n}$, and proper time sequences $\overset{\alpha}{S}$. Further, let $\overset{\alpha}{\overline{r}}=\max_{p\in\gamma[I']}\overset{\alpha}{r}$ and $\overset{\alpha}{\underline{r}}=\min_{p\in\gamma[I']}\overset{\alpha}{r}$, and suppose that $\lim_{\alpha\to\infty}\overset{\alpha}{\overline{r}}/\overset{\alpha}{\underline{r}}=1$. Then, if $\{\overset{\alpha}{d}\}$ is any sequence satisfying $\overset{\alpha}{\underline{r}}\le\overset{\alpha}{d}\le\overset{\alpha}{\overline{r}}$ for all α,*

1. $\lim_{\alpha\to\infty}2\overset{\alpha}{n}\overset{\alpha}{d}=|I'|$, *and*
2. $\limsup_{\alpha\to\infty}\{|(\overset{\alpha}{s}_i-\overset{\alpha}{s}_{i-1})-(\overset{\alpha}{s}_j-\overset{\alpha}{s}_{j-1})|:1\le i,j\le\overset{\alpha}{n}\}=0.$

The proof of each limit equation requires the use of the lemma, but neither depends essentially on steps from the other. The proof of the second limit equation is straightforward, whereas the first proceeds in two parts. The first part derives a preliminary limit equation involving the sum of the arc lengths of segments of $\gamma[I']$ "clocked" by the proper time sequences. The second part shows this sum converges to $|I'|$ as $\alpha\to\infty$.

Proof (First limit equation) Let $\epsilon>0$ be given, and note that I' is compact. Therefore, from the lemma, we have that there is some $\delta>0$ such that for all positive $d<\delta$ and for each $p\in\gamma[I']$ and spacelike unit vector $\rho^a\in T_pM$, $|(s_d-s'_d)/d-2|<\epsilon$, where $s_d>s'_d$ and $\gamma(s'_d)$, $\exp_p(d\rho^a)$, and $\gamma(s_d)$ are distinct points sequentially connected by null geodesics.

Now, the condition that $\lim_{\alpha\to\infty}\overset{\alpha}{\overline{r}}/\overset{\alpha}{\underline{r}}=1$ means that there is some α_0 such that, for all $\alpha>\alpha_0$, $\sqrt{\delta}>(\overset{\alpha}{\overline{r}}/\overset{\alpha}{\underline{r}})-1$, i.e., $\overset{\alpha}{\overline{r}}<\sqrt{\delta}\overset{\alpha}{\underline{r}}+\overset{\alpha}{\underline{r}}$. Since $\lim_{\alpha\to\infty}\overset{\alpha}{r}=0$ for each $p\in\gamma[I']$, we must have as well that $\lim_{\alpha\to\infty}\overset{\alpha}{\underline{r}}=0$, so there is some $\underline{\alpha}_0$ and $\underline{\alpha}'_0$ such

[7]Köhler has proved an analog of the theorem's second limit equation for the special case of geodesics in Weyl spacetimes [19].

that $\overset{\alpha}{\underline{r}} < \sqrt{\delta}/2$ and $\overset{\alpha}{\underline{r}} < \delta/2$ when, respectively, $\alpha > \underline{\alpha}_0$ and $\alpha > \underline{\alpha}'_0$. Hence, if we let $\overset{\alpha}{r}_i$ denote the value of the scalar radius field at $\gamma(\overset{\alpha}{t}_i)$, then for $\alpha > \max\{\alpha_0, \underline{\alpha}_0, \underline{\alpha}'_0\}$, $|(\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1})/\overset{\alpha}{r}_i - 2| < \epsilon$ because $\overset{\alpha}{r}_i \le \overset{\alpha}{\overline{r}} < \delta$ for every $1 \le i \le \overset{\alpha}{n}$. Summing these inequalities yields that

$$\overset{\alpha}{n}\epsilon > \sum_{i=1}^{\overset{\alpha}{n}} \left| \frac{\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}}{\overset{\alpha}{r}_i} - 2 \right| \ge \left| \sum_{i=1}^{\overset{\alpha}{n}} \frac{\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}}{\overset{\alpha}{r}_i} - 2\overset{\alpha}{n} \right|,$$

where the last bound follows from the triangle inequality. Dividing through by $\overset{\alpha}{n}$ gives that

$$\left| \frac{1}{\overset{\alpha}{n}\overset{\alpha}{\overline{r}}} \sum_{i=1}^{\overset{\alpha}{n}} \frac{\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}}{\overset{\alpha}{r}_i/\overset{\alpha}{\overline{r}}} - 2 \right| < \epsilon$$

for every $\alpha > \max\{\alpha_0, \underline{\alpha}_0, \underline{\alpha}'_0\}$, i.e.,

$$\lim_{\alpha\to\infty} \frac{1}{\overset{\alpha}{n}\overset{\alpha}{\overline{r}}} \sum_{i=1}^{\overset{\alpha}{n}} \frac{\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}}{\overset{\alpha}{r}_i/\overset{\alpha}{\overline{r}}} = 2. \tag{7}$$

Now we just need to show that the sum converges to $|I'|$ as $\alpha \to \infty$. We proceed using the squeeze theorem, beginning with the lower bound. Let $b = \max I'$. Noting that we can write

$$\left|I'\right| = b - \overset{\alpha}{s}_{\overset{\alpha}{n}} + \sum_{i=1}^{\overset{\alpha}{n}} (\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}) \ge \sum_{i=1}^{\overset{\alpha}{n}} (\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}), \tag{8}$$

it follows that

$$\begin{aligned} \left|I'\right| - \sum_{i=1}^{\overset{\alpha}{n}} \frac{\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}}{\overset{\alpha}{r}_i/\overset{\alpha}{\overline{r}}} &\ge \sum_{i=1}^{\overset{\alpha}{n}} (\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1})(1 - \overset{\alpha}{\overline{r}}/\overset{\alpha}{r}_i) \\ &\ge (1 - \overset{\alpha}{\overline{r}}/\overset{\alpha}{\underline{r}}) \sum_{i=1}^{\overset{\alpha}{n}} (\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}) \ge (1 - \overset{\alpha}{\overline{r}}/\overset{\alpha}{\underline{r}})\left|I'\right|, \end{aligned} \tag{9}$$

using Eq. (8) in the first and third inequalities and the fact that $\overset{\alpha}{\overline{r}}/\overset{\alpha}{\underline{r}} \ge \overset{\alpha}{\overline{r}}/\overset{\alpha}{r}_i \ge 1$ for each $i \le \overset{\alpha}{n}$ in the second.

For the upper bound, define $\overset{\alpha}{\sigma} \in I'$ as the unique parameter value such that there is some $\overset{\alpha}{\tau} \in I'$ for which both $\gamma(\overset{\alpha}{\sigma})$ and $\gamma(b)$ connect to $\exp_{\gamma(\overset{\alpha}{\tau})}(\overset{\alpha}{r}\overset{\alpha}{\rho}{}^a)$ by null geodesics, where $\overset{\alpha}{\rho}{}^a$ is the orientation vector field for the convergent companion family. By the construction of the proper time sequences, we must have that $\overset{\alpha}{s}_{\overset{\alpha}{n}} \ge \overset{\alpha}{\sigma}$, hence $b - \overset{\alpha}{s}_{\overset{\alpha}{n}} \le$

$b - \overset{\alpha}{\sigma}$. Noting again that $\overset{\alpha}{\overline{r}}/\overset{\alpha}{r}_i \geq 1$ for all $i \leq \overset{\alpha}{n}$, this entails that

$$\left|I'\right| - \sum_{i=1}^{\overset{\alpha}{n}} \frac{\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}}{\overset{\alpha}{r}_i/\overset{\alpha}{\overline{r}}} \leq \left|I'\right| - \sum_{i=1}^{\overset{\alpha}{n}}(\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}) = b - \overset{\alpha}{s}_{\overset{\alpha}{n}} \leq b - \overset{\alpha}{\sigma}. \tag{10}$$

Note further that in the construction of the proper time sequences we fixed $\overset{\alpha}{s}_0 = \min I' = a$, and defined each $\overset{\alpha}{s}_i$ so that $\overset{\alpha}{s}_j > \overset{\alpha}{s}_i$ when $j > i$. We could have equally well fixed $\overset{\alpha}{s}_0 = \max I' = b$, and defined each $\overset{\alpha}{s}_i$ so that $\overset{\alpha}{s}_j < \overset{\alpha}{s}_i$ when $j > i$. Thus we can apply an argument similar to the one at the beginning of this proof to conclude that, for any $\epsilon > 0$, there is a sufficiently large α such that $|(b - \overset{\alpha}{\sigma})/\overset{\alpha}{r}_\tau - 2| < \epsilon$, where $\overset{\alpha}{r}_\tau$ is the value of the scalar radius field at $\gamma(\overset{\alpha}{\tau})$. This implies that $|b - \overset{\alpha}{\sigma}| \leq |b - \overset{\alpha}{\sigma} - 2\overset{\alpha}{r}_\tau| + |2\overset{\alpha}{r}_\tau| < \epsilon \overset{\alpha}{r}_\tau + 2\overset{\alpha}{r}_\tau$ by the triangle inequality. Now, since $\lim_{\alpha\to\infty} \overset{\alpha}{\overline{r}} = 0$ and $0 < \overset{\alpha}{r}_\tau \leq \overset{\alpha}{\overline{r}}$, we must have that $\lim_{\alpha\to\infty} \overset{\alpha}{r}_\tau = 0$, hence $\lim_{\alpha\to\infty} \overset{\alpha}{\sigma} = b$. Since $\lim_{\alpha\to\infty} \overset{\alpha}{\overline{r}}/\overset{\alpha}{\underline{r}} = 1$ as well, Eqs. (9) and (10) entail that

$$\lim_{\alpha\to\infty} \sum_{i=1}^{\overset{\alpha}{n}} \frac{\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}}{\overset{\alpha}{r}_i/\overset{\alpha}{\overline{r}}} = \left|I'\right|. \tag{11}$$

Combining Eqs. (7) and (11) gives

$$2 = \lim_{\alpha\to\infty} \frac{1}{\overset{\alpha}{n}\overset{\alpha}{\overline{r}}} \sum_{i=1}^{\overset{\alpha}{n}} \frac{\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}}{\overset{\alpha}{r}_i/\overset{\alpha}{\overline{r}}} = \left(\lim_{\alpha\to\infty} \frac{1}{\overset{\alpha}{n}\overset{\alpha}{\overline{r}}}\right)\left|I'\right|,$$

which, after rearranging, yields that $\lim_{\alpha\to\infty} 2\overset{\alpha}{n}\overset{\alpha}{\overline{r}} = |I'|$. Since $\lim_{\alpha\to\infty} \overset{\alpha}{\overline{r}}/\overset{\alpha}{\underline{r}} = 1$, we must have $\lim_{\alpha\to\infty} \overset{\alpha}{d}/\overset{\alpha}{\overline{r}} = 1$ as well. Therefore

$$\lim_{\alpha\to\infty} 2\overset{\alpha}{n}\overset{\alpha}{d} = \left(\lim_{\alpha\to\infty} 2\overset{\alpha}{n}\overset{\alpha}{\overline{r}}\right)\left(\lim_{\alpha\to\infty} \overset{\alpha}{d}/\overset{\alpha}{\overline{r}}\right) = \left|I'\right|. \qquad \square$$

Proof (Second limit equation) For the second part of the theorem, we begin similarly: given some $\epsilon > 0$, for large enough α, $|\overset{\alpha}{\Delta}_i/\overset{\alpha}{r}_i - 2| < \epsilon/3|I'|$ for every $1 \leq i \leq \overset{\alpha}{n}$, where $\overset{\alpha}{\Delta}_i = \overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}$. Put $U_\alpha = \arg\max_{1\leq i\leq \overset{\alpha}{n}}(\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1})$ and $L_\alpha = \arg\min_{1\leq i\leq \overset{\alpha}{n}}(\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1})$, letting $\overset{\alpha}{r}_{U_\alpha}$ and $\overset{\alpha}{r}_{L_\alpha}$ be the values of the scalar radius field at $\gamma(\overset{\alpha}{t}_{U_\alpha})$ and $\gamma(\overset{\alpha}{t}_{L_\alpha})$, respectively. Then

$$\frac{2\epsilon}{3|I'|} > \left|\frac{\overset{\alpha}{\Delta}_{U_\alpha}}{\overset{\alpha}{r}_{U_\alpha}} - 2\right| + \left|2 - \frac{\overset{\alpha}{\Delta}_{L_\alpha}}{\overset{\alpha}{r}_{L_\alpha}}\right| \geq \left|\frac{\overset{\alpha}{\Delta}_{U_\alpha}}{\overset{\alpha}{r}_{U_\alpha}} - \frac{\overset{\alpha}{\Delta}_{L_\alpha}}{\overset{\alpha}{r}_{L_\alpha}}\right| \geq \frac{\overset{\alpha}{\Delta}_{U_\alpha}}{\overset{\alpha}{r}_{U_\alpha}} - \frac{\overset{\alpha}{\Delta}_{L_\alpha}}{\overset{\alpha}{r}_{L_\alpha}}$$

by the triangle inequality, and multiplying through by $\overset{\alpha}{r}_{U_\alpha}$ gives

$$2\epsilon\overset{\alpha}{r}_{U_\alpha}/3\left|I'\right| > \overset{\alpha}{\Delta}_{U_\alpha} - \overset{\alpha}{\Delta}_{L_\alpha}\overset{\alpha}{r}_{U_\alpha}/\overset{\alpha}{r}_{L_\alpha}. \tag{12}$$

But note that, by the assumption of the theorem, $\overset{\alpha}{\overline{r}}/\overset{\alpha}{\underline{r}} \to 1$ as $\alpha \to \infty$, so for sufficiently large α we have that $\overset{\alpha}{r}_{U_\alpha}/\overset{\alpha}{r}_{L_\alpha} \leq \overset{\alpha}{\overline{r}}/\overset{\alpha}{\underline{r}} < 1 + \epsilon/3|I'|$. Hence from Eq. (12) we have $2\epsilon\overset{\alpha}{r}_{U_\alpha}/3|I'| > \overset{\alpha}{\Delta}_{U_\alpha} - \overset{\alpha}{\Delta}_{L_\alpha}(1 + \epsilon/3|I'|)$ and

$$(2\overset{\alpha}{r}_{U_\alpha} + \overset{\alpha}{\Delta}_{L_\alpha})\epsilon/3\left|I'\right| > \overset{\alpha}{\Delta}_{U_\alpha} - \overset{\alpha}{\Delta}_{L_\alpha} = \sup\left\{\left|(\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}) - (\overset{\alpha}{s}_j - \overset{\alpha}{s}_{j-1})\right| : 1 \leq i, j \leq \overset{\alpha}{n}\right\}.$$

Clearly $\overset{\alpha}{\Delta}_{L_\alpha} \leq |I'|$ in general, and $\overset{\alpha}{r}_{U_\alpha} \leq |I'|$ for sufficiently large α, so we have at last that

$$\epsilon > \sup\left\{\left|(\overset{\alpha}{s}_i - \overset{\alpha}{s}_{i-1}) - (\overset{\alpha}{s}_j - \overset{\alpha}{s}_{j-1})\right| : 1 \leq i, j \leq \overset{\alpha}{n}\right\}$$

for sufficiently large α. □

3 Discussion

The companion curves represent a sequence of possible light clocks that measure the length of $\gamma[I']$, where there is no restriction placed on their acceleration or the spacetime in which γ is embedded. Instead of limiting attention to light clocks with an equilibrium state, as Maudlin's argument did, one can instead focus on light clocks that are, in a certain sense, sufficiently small and unvarying as determined by their scalar radius field r. Before interpreting this latter condition, it is important to note that the scalar radius field r is not a privileged measure of the distance between the two mirrors of the light clock. One could very well pick some other spacelike vector field on γ and some other scalar parameter to trace out the same companion curve, and this new pair would bear a systematic functional relationship to ρ^a and r. The constraints on r, as determined by the theorem, would then fix constraints on this new scalar parameter. But this would yield an entirely equivalent construction, as far as the spacetime geometry is concerned. Specifying r and the orientation vector field ρ^a completely determines the location of the companion curve, hence of the trajectory through spacetime of the light clock's second mirror.

Having a *convergent* family of companion curves means that there is always available a sufficiently "small" light clock as determined by the scalar radius field r. (In fact, requiring the condition that $\lim_{\alpha\to\infty}\overset{\alpha}{\overline{r}}/\overset{\alpha}{\underline{r}} = 1$ in the theorem further restricts attention to *uniformly* convergent companion families, in the sense that $\overset{\alpha}{r} \to 0$ uniformly as $\alpha \to 0$.) It is equivalent to the condition that $\lim_{\alpha\to\infty}(\log\overset{\alpha}{\overline{r}} - \log\overset{\alpha}{\underline{r}}) = 0$, which one can interpret as requiring that, for sufficiently small light clocks, the order of magnitude of variation in the range of the scalar radius field is small. This makes sense, for if the field's maximum $\overline{r}$ and minimum $\underline{r}$ are not of the same scale, one would expect that the error induced by variation in r is never reduced.

One can think of the role of the parameter d as follows. To apply the theorem, one does not need detailed knowledge of exactly how a light clock will expand and contract, only what the range of its corresponding scalar radius field will be. One can then pick d as some fixed value in that range. Then the first limiting equation, concerning the accuracy of the light clock, states that (under the conditions of the theorem) twice the product of this d with the number of observed bounces will approximately equal the proper time elapsed.[8] The second limiting equation, concerning the regularity of the clock, states that the maximum difference in elapsed time between any two ticks over the course of the clock's run will be small.

With that interpretive gloss, one can then state the theorem in words as follows. Given a closed segment of a timelike curve and any $\epsilon_A, \epsilon_R > 0$, there is a sufficiently small and unvarying light clock that measures the proper time along that segment within an accuracy of ϵ_A and ticks with no more than ϵ_R variation in regularity. Compared to Maudlin's argument, the existence of an equilibrium state has been replaced with requirements on the value and range of variation of the scalar radius field r, while the restriction to Minkowski spacetime and inertial motion has been dropped. Compared to the work of Gautreau and Anderson, the restriction to Born rigid, negligibly accelerating clocks has been dropped.

Of course, one need not be satisfied merely proving the existence of such light clocks. If one assumes the curve γ belongs to a higher-order continuity class than just $C^{(2)}$, one should be able to examine accordingly higher-order series expansions of the world-function to obtain a more quantitative estimate of a given light clock's error. In particular, future work can use the techniques developed in Ch. III, Sect. 8 of Synge [16] for this purpose. One could also study the converse problem, as Kowalski [13] did for certain cases in Minkowski spacetime: given a light clock with certain characteristics, under what circumstances of acceleration and curvature will it be approximately ideal? Since the attractiveness of studying light clocks arose from the fact that one can abstract away from most of dynamical complications of particular matter fields, some care will be needed in demarcating what the appropriate characteristics might be.

Lastly, there are other generalizations of the theorem that may be of interest. In both the lemma and the theorem, it is clear that one can just as easily use timelike instead of null geodesics as the clock mechanism, i.e., bouncing massive particles instead of photons. But one must additionally require that the speed v of the particles relative to the worldline whose proper time is to be measured be constant.[9] Formally, if μ^a and ξ^a represent, respectively, the tangent vectors for these particles and the worldline γ, this is the requirement that $\mu^a \xi_a = (1-v)^{-1/2}$ is constant on

[8] If one is interested in results of a more operational flavor [6], then suppose one can survey segments $\gamma_A[I'_A]$ and $\gamma_B[I'_B]$ of two timelike worldlines with light clocks whose scalar radius fields have similar ranges. Putting $\overset{\alpha}{n}_A$ and $\overset{\alpha}{n}_B$ as the bounce numbers for the two respective clocks, we have that $\lim_{\alpha\to\infty} \overset{\alpha}{n}_A/\overset{\alpha}{n}_B = |I'_A|/|I'_B|$. That is, the ratio of bounce numbers gives in the limit the ratio of proper times of the two segments.

[9] In fact, one should be able to relax this requirement so that the same kind of limiting relation holds if v is not constant but not varying too much, just as the theorem does with the light clock's radial distance field r.

(the image of) γ. The only change in the theorem is that the first limit equation becomes $\lim_{\alpha\to\infty} 2\overset{\alpha}{n}\overset{\alpha}{d}/v = |I'|$. Thus one could apply the same method to geometrized Newtonian theory (also known as Newton–Cartan theory),[10] where one cannot avail oneself of the standard of constancy induced by the null cones.

Acknowledgements Thanks to Erik Curiel for bringing the work of Gautreau and Anderson to my attention, and to David Malament similarly for the work of Synge and for suggestions leading to the formulation of the lemma involved in the proof of the central theorem. This work was supported by a National Science Foundation Graduate Research Fellowship.

References

1. Malament, D.B.: Topics in the Foundations of General Relativity and Newtonian Gravitation Theory. University of Chicago Press, Chicago (2012)
2. Bailey, J., et al.: Measurements of relativistic time dilation for positive and negative muons in a circular orbit. Nature **268**, 301–305 (1977)
3. Mainwaring, S.R., Stedman, G.E.: Accelerated clock principles in special relativity. Phys. Rev. A **47.5**, 3611–3619 (1993)
4. Mashhoon, B.: Measurement theory and general relativity. In: Hehl, F.W., Kiefer, C., Metzler, R.J.K. (eds.) Black Holes: Theory and Observation, pp. 269–284. Springer, Berlin (1998)
5. Marzke, R.F.: The Theory of Measurement in General Relativity. AB thesis, Princeton University (1959)
6. Marzke, R.F., Wheeler, J.A.: Gravitation as geometry—I: the geometry of space-time and the geometrodynamical standard meter. In: Chiu, H., Hoffman, W.F. (eds.) Gravitation and Relativity, pp. 40–64. Benjamin, New York (1964)
7. Møller, C.: Old problems in the general theory of relativity viewed from a new angle. Dan. Mat. Fys. Medd. **30.10** (1955)
8. Møller, C.: The ideal standard clocks in the general theory of relativity. Helv. Phys. Acta, Suppl. **4**, 54–57 (1956)
9. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. Freeman, San Francisco (1973)
10. Eisele, A.M.: On the behaviour of an accelerated clock. Helv. Phys. Acta **60**, 1024–1037 (1987)
11. Maudlin, T.: Philosophy of Physics: Space and Time. Princeton University Press, Princeton (2012)
12. Marder, L.: Time and the Space-Traveller. University of Pennsylvania Press, Philadelphia (1971)
13. Kowalski, F.V.: Accelerating light clocks. Phys. Rev. A **53.6**, 3761–3766 (1996)
14. Gautreau, R.T.: Operational Approach to Space-Time Measurements in Relativity. Ph.D. dissertation, Stevens Institute of Technology (1966)
15. Anderson, J.L., Gautreau, R.: Operational formulation of the principle of equivalence. Phys. Rev. **185.5**, 1656–1661 (1969)
16. Synge, J.L.: Relativity: the General Theory. North-Holland, Amsterdam (1960)
17. Sachs, R.K., Wu, H.: General Relativity for Mathematicians. Springer, New York (1977)
18. O'Neill, B.: Semi-Riemannian Geometry. Academic Press, San Diego (1983)
19. Köhler, E.: Measurement of proper time in a weyl space. Gen. Relativ. Gravit. **9.11**, 953–959 (1978)

[10] See Malament [1, Ch. 4] for an introduction.