Similarity Structure and Diachronic Emergence

Samuel C. Fletcher*

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Abstract

I provide a formally precise account of diachronic emergence of properties as described within scientific theories, extending a recent account of synchronic emergence using *similarity structure* on the theories' models. This similarity structure approach to emergent properties unifies the synchronic and diachronic types by revealing that they only differ in how they delineate the domains of application of theories. This allows it to apply also to cases where the synchronic/diachronic distinction is unclear, such as spacetime emergence from theories of quantum gravity. In addition, I discuss two further case studies—finite periodicity in van der Pol oscillators and two-dimensional quasiparticles in the fractional quantum Hall effect—to facilitate comparison of this approach to others in the literature on concepts of emergence applicable to the sciences. My discussion of the fractional quantum Hall effect in particular may be of independent interest to philosophers of physics concerned with its interpretation.

1 Introduction and Motivation

In science, emergence can unfold in time. As fluid flow in a pipe increases, "puffs" of turbulence emerge, arising and dying out much like the idealized organisms in an ecological predator-prey model (Shih et al., 2016). Groups of animals, such as herring, starlings, and zebra, can coordinate swarming behavior, with emergent simple collective movement rules despite a lack of central direction or command (Wood and Ackland, 2007). Some neuroscientists argue that the computational properties and capacity for learning of neural networks emerge from the process of their mutual interaction (Hopfield, 1982). And diverse communities of people can emerge from dynamic interactions of heterogeneous individuals in social networks (Han et al., 2017), as can other forms of cooperation and organization as

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individuals adjust their interaction strategies over time (Challet and Zhang, 1997). In each of these cases,[1] seemingly novel high-level properties arise as low-level dynamics proceeds.

Despite the clear importance and wide relevance of *diachronic emergence*, scientists have generally not attempted to give a definition or a precise account of emergence as a process. Emergence being a trans-disciplinary concept, this is perhaps understandable; it is thus an attractive target of theorizing for philosophers of science. But neither have most philosophers taken up this goal, as they have typically concerned themselves instead with a *synchronic* version, according to which some (typically higher-level) properties or entities are, at some time, emergent with respect some others (typically residing at a lower level). (Exceptions include Humphreys (1997, 2016b), Rueger (2000), McGivern and Rueger (2010), and Guay and Sartenaer (2016, 2018), whose accounts I discuss in section 4.) Moreover, most philosophical discussions of emergence, drawing from metaphysics and philosophy of mind, have unclear application to that concept's use in the sciences.

I attempt to rectify this situation in section 2 by providing a formally precise account of diachronic emergence of properties as described within scientific theories. I intend both to capture scientists' usage of the diachronic emergence concept and to provide a fruitful framework to explore, adjudicate, and describe future putative cases thereof. To do so, I extend my recent account of synchronic emergence (Fletcher, 2020) that analyzes emergence in terms of *comparative novelty*, making novelty precise with *similarity structure*, which is a kind of generalization of topological (or uniform) structure. This is a structure on the set of models of the scientific theories under comparison that encodes relevant, qualitative ways in which the models are similar; comparative novelty is then just comparative dissimilarity. Because, as I show, such dissimilarity comes in graded degrees, I distinguish four different types of emergence—weak, strong, non-reductive, and radical—partially ordered in strength, mirroring those for synchronic emergence.[2]

One consequence of how I achieve this extension is that the differences between synchronic and diachronic emergence arise only from the role that time plays in delineating domains of application. As phenomena change, they can fall into (or out of) a domain of application, so that the novel applicability of a theory or model makes diachronic emergence possible when the theory or model ascribes novel properties to the phenomenon. But the same could be said, e.g., of phenomena at different spatial locations, or examined at different levels of precision or with different tools. In scientific contexts, this distinction therefore functions only to delineate the *domains of application* of different scientific theories or models. This

[1] Each of the mentioned subjects has an enormous associated literature; the given citations are merely representative.

[2] In fact, much of this section, except for the exposition of and comparison with diachronic emergence, recapitulates the relevant portion of Fletcher (2020, §2–3).

permits two advantages for the present account of emergence. First, it allows a unified account of synchronic and diachronic emergence, whereas such concepts are often regarded as conceptually distinct (Humphreys, 2008, 2016a).[3] Second, it applies to cases where time itself, or temporal duration, is an emergent property, as may occur for some theories of quantum gravity; the synchronic/diachronic distinction cannot clearly apply when one of the theories in whose models the putative emergent property resides cannot be related temporally to another.

I briefly discuss these advantages in the concluding section 5. Before doing so, I provide in section 3 two other extended example applications to illustrate the similarity structure approach to diachronic emergence. These, in sections 3.1 and 3.2, concern respectively emergent finite periodicity in the van der Pol oscillator, and two-dimensional quasiparticles in the fractional quantum Hall effect (FQHE). These examples facilitate comparison with other accounts of diachronic emergence in section 4, as they are taken up by the authors under discussion: Rueger (2000) and McGivern and Rueger (2010) take up dynamical systems, while Guay and Sartenaer (2016, 2018) take up the FQHE. There, I shall argue that my account of diachronic emergence based on similarity structure is more general and better captures the scientific details of these examples. Readers interested in the interpretation of the FQHE may find my discussion thereof in sections 3.2 and 4.2 of independent interest.

Before proceeding, I have two remarks on the scope of this essay. First, as will be evident from the development in section 2, the similarity structure approach to diachronic emergence makes emergence of properties precise to the extent that the scientific models in which (representations of) these properties reside can be made precise. This is why, to illustrate this precision in section 3, I have focused on examples from physics, despite the multidisciplinary litany with which I began this section. There is nothing in this approach specifically attuned to the content of theories and models in physics, so I expect it to apply to examples of diachronic emergence from other sorts of theories and models.

Second, as there are many distinctions between different sub-types of emergence (beyond the synchronic and diachronic), it is worth remarking on how these distinctions overlay the account of diachronic emergence (and emergence more generally) given through similarity structure. Emergence can in general pertain to entities, disciplines, levels, and other objects, but the present account concerns properties as represented within models of scientific theories.[4] Following Humphreys (2016a,b), emergence can also be epistemological, ontological,

[3] Despite this distinction, Humphreys (2008, pp. 431–2) was "optimistic that we shall eventually find a unifying framework that explains why synchronic and diachronic emergence both count as emergence in some more general sense."

[4] In this sense, the present account has affinities with what Teller (2010) calls "story-2" about the often imperfect representational relationship between scientific theories and the world. Unlike Teller, however,

or conceptual, where these categories may overlap. Emergent properties in the similarity structure approach are epistemologically emergent, in the sense that their comparative novelty is associated with a failure of deducibility from the models of the comparison theory. (I will describe the nature of this failure in section 2.) Whether they count as ontologically emergent will depend on the strength of the property's realist credentials, an important issue that I will nevertheless not further address. Whether they count as conceptually emergent depends on whether the property is in some sense definable within the comparison theory.

2 Similarity Structure and Emergence

2.1 Similarity Structure

The similarity structure approach to emergence in general starts with an insight from Butterfield (2011a,b, 2014), that an emergent property is one that is novel with respect to some comparison class,[5] formalizing it using a collection of qualitative, binary relations.

Definition 1. A *similarity relation* $\sim$ on a set X is a non-empty binary relation on X that is *quasi-reflexive*: for all $x, y \in X$, if $y \sim x$, then $y \sim y$ and $x \sim x$.

The relation $y \sim x$ is interpreted as "y is similar to x" in a respect determined by the extension of the relation. For example, two models of harmonically oscillating bobs hanging from a spring could be similar in virtue of having the same mass parameter or the same period of oscillation up and down. These simple examples involve similarity relations that are not just quasi-reflexive, but *reflexive*—$x \sim x$ for every $x \in X$—and *symmetric*: if $y \sim x$ then $x \sim y$. Many have imposed these stronger conditions on similarity relations (Carnap, 1967; Schreider, 1975; Mormann, 1996; Konikowska, 1997), but subtler examples betray them (Fletcher, 2020, ming) and they are not needed in what follows in any case.[6]

Just as I supposed that there were at least two contextually relevant ways above in which harmonically oscillating bobs are similar, one can consider arbitrarily many similarity relations on a set X to capture comparisons of similarity in multiple respects.

Definition 2. A *similarity space* is an ordered pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a non-empty set of similarity relations on X, called a *similarity structure*.

I remain agnostic about the ontological implications of this sort of emergence (or the lack thereof), as I emphasize in the sequel.

[5] In some writings, Butterfield (2011b, p. 1066) requires that if a property is emergent, it is also "robust," but does not in others (Butterfield, 2014). I follow the later formulations for reasons I adumbrate in Fletcher (2020, §3).

[6] For an example of the failure of full reflexivity, consider the set of all people, and suppose that they are similar to one another to the extent that they have similar professional baseball RBI averages. I'm not similar to myself in this respect because I've never played professional baseball.

For quantitative properties like the masses and periods of bobs, for example, one can define a similarity relation for each positive ϵ that relates a pair of models in case the value of their quantitative property (mass or period) differs by less than ϵ. Explicitly, in the case of mass, let $\sim_\epsilon$ be the relation on the collection of models of harmonic oscillators such that $x \sim_\epsilon y$ just in case the mass of the bob in models x and y are within ϵ grams of each other. The collection of all such $\sim_\epsilon$, for $\epsilon \geq 0$, is a similarity structure $\mathcal{S}$. If the mass is 1g for x and 1.1g for y, then $x \sim_\epsilon y$ for $\epsilon \geq .1$ but $x \nsim_\epsilon y$ for $\epsilon < .1$. Any models with bobs of the *same* mass will then be *arbitrarily* similar to each other according to this similarity structure, in the sense that they are related to each other by *each* similarity relation in that structure.

To define this more formally, let $\wp X$ be the power set of X, i.e., the set of all subsets of X.

Definition 3. The *closeness operator* $\mathrm{cl}_\mathcal{S} : \wp X \to \wp X$ of a similarity space $(X, \mathcal{S})$ is defined by $\mathrm{cl}_\mathcal{S}(A) = \{x \in X : \forall \sim \in \mathcal{S}, \exists a \in A : a \sim x\}$.

The value of the closeness operator acting on a set $A \subseteq X$ is the collection of all elements of X that are arbitrarily similar to some element in A, i.e., those similar to an element of A in all respects determined to be relevant by $\mathcal{S}$. Because similarity relations are quasi-reflexive, this will always include A, i.e., the closeness operator is extensive in the subset ordering.[7] So, suppose A is some nonempty subset of the collection of models X representing the harmonically oscillating bobs and $\mathcal{S}$ is the similarity structure from just before. Then $\mathrm{cl}_\mathcal{S}(A)$ will contain all the models of harmonically oscillating bobs with the same mass as any such model in A. For instance, if A consists of one model for each mass strictly greater than 1g, then $\mathrm{cl}_\mathcal{S}(A)$ will contain *all* the models whose bobs have mass at least 1g (regardless of their oscillatory features or other properties).

There is no restriction on what sorts of things are collected in X. In addition to objects and states of affairs, they can also be values of a property.[8]

Definition 4. A *property assignment* (or *valuation*) on a collection X is a (perhaps partial) map $\nu : X \nrightarrow V$, where V is called its space of *(property) values.* Further, if $x \in X$ is (not) in the domain of ν, then the property represented by ν is said to be *(not) defined for x.*

[7] In fact it is in general a preclosure operator instead of a closure operator because it is not necessarily idempotent—i.e., it is not generally the case that $\mathrm{cl}_\mathcal{S}(A) = \mathrm{cl}_\mathcal{S}(\mathrm{cl}_\mathcal{S}(A))$. See **?**Fletcher (ming) for more on the comparison of similarity spaces with topological spaces.

[8] In addition, realistic models of scientific theories are typically themselves highly structured, with sub-objects, relations among them, and various other properties and pieces of architecture. Instead of representing properties of sub-objects and such with tedious compositions of property assignments with maps picking out the relevant sub-objects, in what follows I will model them simply as property assignments to the models themselves. In such cases, it just needs to be clear that the sub-object in question is in fact well-defined.

A property assignment represents a property intensionally, as a map from models to values. Quantitative properties, such as mass and period, have some subset of the real numbers as their values. The property assignment for mass in particular is given by a map of the form $\nu_m : X \to (0, \infty)$. Propositional properties, meanwhile, have the Boolean domain $\{\top, \bot\}$ as their space of values. For an example, consider the property of *being in harmonic motion*: all models of the oscillating bob in motion will have this property (i.e., value $\top$). The models representing a bob at rest will not (i.e., they will have value $\bot$).

As I alluded above, these values can be similar to one another just like the models representing the harmonically oscillating bobs can. In fact, when I defined the similarity structure $\mathcal{S}$ on the models of the oscillating bobs, I took advantage already of a plausible similarity structure on the values of the mass parameter, that consisting of all similarity relations $\sim_\epsilon^m$, for $\epsilon \geq 0$, relating two mass parameters if and only if they differ by no more than ϵ. One natural qualitative case is the similarity structure on the Boolean domain $\{\top, \bot\}$ consisting of a single similiarity relation whose extension is $\{(\top, \top), (\bot, \bot)\}$, according to which "true" and "false" are each only similar to themselves.

2.2 Synchronic and Diachronic Emergence

The account of emergent properties in terms of similarity structure—in both the synchronic and diachronic cases—has a formal part and an informal part. The formal part describes the relations between property assignment values, models, and similarity structure necessary for the value of a property assignment to a model or collection of models to be emergent with respect to a comparison class of models. In particular, emergence pertains to values of a property assignment that obtain in one or more models of some theory, in comparison with some other models of some theory. (Many discussions of emergent properties restrict attention to a propositional property assignment to a single model, but the account here applies more generally to sets of property values for sets of models.) It thus yields not a single emergence concept, but a collection of emergence concepts partially ordered by logical implication. The informal part of the account demands that certain conditions of application be fulfilled about what the formal objects represent. Differences in this informal part will delineate synchronic from diachronic emergence.

To begin with the formal part: let X be a collection of scientific models equipped with a similarity structure $\mathcal{S}$ and let V be the space of property values with a similarity structure $\mathcal{V}$ for some property assignment $\nu : X \nrightarrow V$. The collection X includes the models of both of the theories under consideration: those with putative emergent values of a property, and those from the comparison class—the ones with respect to which the values are emergent.

Definition 5. With respect to the models $A \subseteq X$, the values $\nu[B]$ of a collection of models $B \subseteq X$ under the assignment ν are said to be:

1. *weakly emergent* when $\nu[B] \nsubseteq \nu[A]$;
2. *strongly emergent* when $\nu[B] \nsubseteq \mathrm{cl}_{\mathcal{V}}(\nu[A])$;
3. *non-reductively emergent* when $\nu[B] \nsubseteq \nu[\mathrm{cl}_{\mathcal{S}}(A)]$; and
4. *radically emergent* when $\nu[B] \nsubseteq \mathrm{cl}_{\mathcal{V}}(\nu[\mathrm{cl}_{\mathcal{S}}(A)])$.[9]

For each of these, it may be said to obtain *completely* when the sets related by the non-inclusion are non-intersecting. Otherwise, it may be said to obtain *partially.*

Weak emergence formally requires only the mere non-inclusion of the values of the property in question, $\nu[B]$, with those for the models in the comparison class, $\nu[A]$. To illustrate by continuing the example of the harmonically oscillating bobs from the previous subsection, let $\nu : X \to [0, \infty)$ denote the property assignment of the oscillatory period of the bobs. Then if A is the collection of models whose periods are non-zero and B those whose periods are zero (i.e., those that are not moving harmonically), then the zero-period property is emergent with respect to the nonzero-period property.

Clearly, no unexpectedness or comparative unexplicability necessarily accompanies mere non-inclusion. These stronger criteria are formalized above along two dimensions, each captured by strong and non-reductive emergence. The strong emergence of the values of a property of models B requires that they also be not sufficiently similar to the values for the models of the comparison class A—they are unexpected because they are not even similar to the values available for consideration from the comparison class. This requires a similarity structure $\mathcal{V}$ on the space of values that the property can take on. For instance, consider the Boolean values of the "being in harmonic motion" property for the harmonically oscillating bob models with the similarity structure for them descibed in the last subsection: "true" is just similar to "true" and "false" to "false." Not being in motion will be strongly emergent in models in which the bob is at rest with respect to models in which the bob is in harmonic motion.

Of course, there are clearly ways in which the *models* with the bob at rest are similar to models with the bob in harmonic motion, for arbitrarily slight such motions. This other

[9] One can in fact define infinitely many concepts of emergence based on iterations of the closeness operators: with respect to the models $A \subseteq X$, the values $\nu[B]$ of a collection of models $B \subseteq X$ can be said to be (n, m)-*emergent* when $\nu[B] \nsubseteq \mathrm{cl}^n_{\mathcal{V}}(\nu[\mathrm{cl}^m_{\mathcal{S}}(A)])$. However, if the similarity structures on the spaces of models and property values have idempotent closeness operators, then the above four concepts are at most the ones that are distinct.

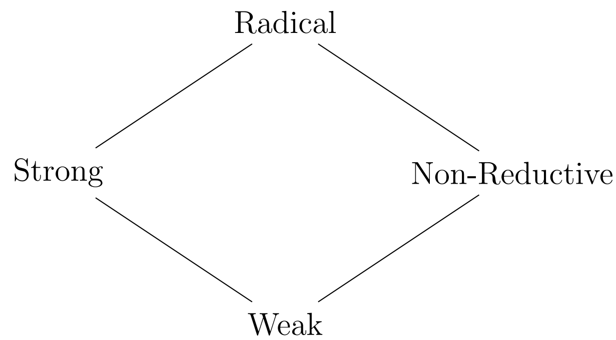


Figure 1: A Hasse diagram of the types of emergence, where the partial order between them is logical entailment.

way of formalizing dissimilarity results in non-reductive emergence: the values of a property assignment to models B are non-reductively emergent with respect to those of the comparison class A just when they are not included even among those arbitrarily similar to those of A. This requires a similarity structure $\mathcal{S}$ on the joint collection of models. Consider such a structure for the models of harmonically oscillating bobs just like that in the previous subsection, except based on oscillatory periods instead of masses: two models will be related by $\sim_\epsilon$ just in case their periods are within ϵ of one another, for each $\epsilon \geq 0$. In this case, not being in harmonic motion will *not* be non-reductively emergent in models in which the bob is at rest with respect to models in which the bob is in harmoic motion because those at rest are arbitrarily similar to those in motion. But it will be so when the comparison class consists of all models with periods greater than δ for some $\delta > 0$, as the models arbitrarily similar to those are just the ones with periods equal to or greater than δ.

Finally, one can combine the criteria for strong and reductive emergence to yield radical emergence: the emergent values of the property are not even arbitrarily similar to those of the models arbitrarily similar to those in the comparison class. Accordingly, this requires similarity structure on both the space of values that a property can take on and the joint collection of models themselves. Continuing the immediately preceding example, we can add the similarity structure on the Boolean values of the "being in motion" property previously discussed. The results are the same: for the comparison class of models with periods greater than δ, not being in harmoic motion is radically emergent in the zero-period models. (For more examples of these types of emergence, see Fletcher (2020).)

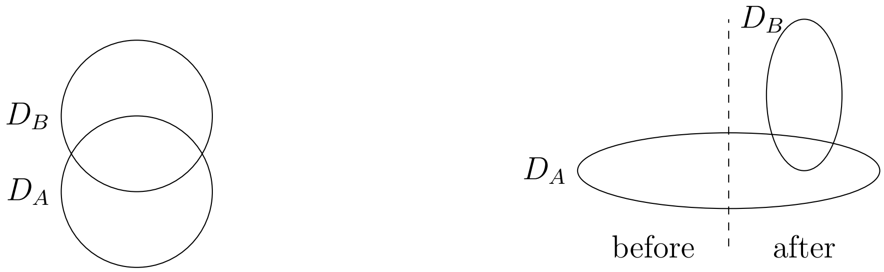
Each of these various sorts of novelty for the emergent values of the property assignment to B entails a sort of lack of deducibility from the models in A, the hallmark of epistemic emergence. When a property with emergent values for models in B is not even defined for models in A, those values would count as conceptually emergent, too. Whether they are also ontologically emergent depends, again, on the property values' realist credentials. And, each type of emergence bears logical relations to the others, as in figure 1. Radical emergence entails both non-reductive and strong emergence, each of which entails weak emergence. None of the converse entailments is true in general, nor is any entailment relation between strong and non-reductive emergence.

In order to draw any of these conclusion from the formal features of the scientific models, however, the appropriate informal conditions of application must be satisfied, of which there are two. First, the models of the comparison class A should be *at least as fundamental* as those in which the putative emergent property value resides. With this condition, weak emergence requires more than mere non-inclusion, as it captures the idea that emergent properties must "arise" (whether synchronically or diachronically) from a base that is no more fundamental. Like with the ontological character of emergence, I remain agnostic here about which account of relative fundamentality to employ, in part because the sort needed will depend on the aims of the models' representations. Some realist-oriented possibilities include supervenience or composition between different scales or levels of description, reality, or explanation (List, 2019). (See Tahko (2018) for further metaphysical possibilities.) An anti-realist-oriented possibility could be empirical adequacy (van Fraassen, 1980): one set of models is at least as fundamental as another when it is at least as empirically adequate for at least the phenomena under consideration.

This informal condition of application is common to both synchronic and diachronic emergence. These two types differ slightly on the second informal condition of application; this in fact constitutes the basis of the distinction between the two. This condition is that the models in A and B should have *overlapping domains of application*. (This is not a formal requirement, as the structure of a model itself does not determine what the model represents.) For a given representational purpose circumscribed by allowable imprecision, abstractions, and idealization, in general many models of a theory will be adequate to represent some phenomenon—e.g., in differing only by small values of some parameter. Consequently, it is possible for models of different theories to be applied to the same phenomena; some of these models in turn may then ascribe different, novel properties.

For synchronic emergence, the elements of a subset of both the models of A and B are adequate to represent the same phenomenon. This phenomenon may be defined at a point of time or over a range of time; what makes this *synchronic* emergence is that the times at which the models of A and B have overlapping domains of application are the same for the putatively emergent value of the property in question—see figure 2a.

By contrast, for diachronic emergence, the models of A have a distinctly wider temporal domain in which they are representationally adequate. This temporal domain, which is



(a) Synchronic emergence (b) Diachronic emergence

Figure 2: Diagrammatic comparison of synchronic and diachronic emergence according to the similarity structure approach. Synchronic emergence requires only that the models of B and A—the latter of which is at least as fundamental as the former—have overlapping domains of application, denoted by D_B and D_A, respectively. Diachronic emergence requires that the domain of application of A be wider temporally than the domain of application of B.

defined not externally but through the theory whose models include those in A, is divided into two periods, "before" and "after," as depicted in figure 2b. Often, a collection of models of A will be representationally adequate during both the "before" and "after" periods, while only in the latter "after" period do the models of B become representationally adequate. If these models ascribe novel properties, then one can say that they have diachronically emerged from or relative to A (according to one of the types of emergence described above).

In the similarity structure approach to emergence, synchronic and diachronic emergence are quite similar to one another. Their types and conditions of application, both formal and informal, are the same, save for the way in which the models ascribing the putative emergent property share, or are adequate for, a domain of application common with the models from which they emerge. In both cases, emergence can be significant because it signals to scientists when the application of a new theory will be fruitful, if the emergent property is scientifically interesting. In particular, models with emergent properties can facilitate simplified scientific explanations and spur the pursuit of new directions for research.

3 Applications

In this section I present two examples of diachronic emergence using the similiarity structure approach: finite oscillatory periods (even harmonic ones) from the van der Pol oscillator (section 3.1) and two-dimensional quasiparticles in the context of the FQHE from non-relativistic quantum field theory (section 3.2). Each of these involves the emergence of a novel property value from a theory that seems to be incompatible with it, and I discuss what sorts of similarity structures on models and spaces of property values yield what sorts of conclusions about the type of resultant emergence. Each example also is rich enough to be presented in significant detail; instead, I will elaborate only the minimal scientific details needed to illustrate the similarity structure approach, facilitate its comparison in section 4 with other accounts of diachronic emergence that discuss some of these examples, and point towards future applications regarding the emergence of spacetime in quantum gravity in section 5.[10] (Where appropriate, I will direct the reader to the relevant research literature for more extensive reviews.)

3.1 Finite Periods in the van der Pol Oscillator

The van der Pol oscillator (1926), originally described in the context of early electrical circuits employing vacuum tubes, is a one-dimensional dynamical system similar to a harmonic oscillator, but with a non-linear damping force. Its equation of motion, in dynamical variable x, may be expressed as

$$\ddot{x} + \eta(x^2 - a^2)\dot{x} + \omega^2 x = 0, \tag{1}$$

where the overdot represents differentiation with respect to time, and η, a, and ω are positive real parameters describing the damping magnitude, critical displacement, and natural angular frequency, respectively. The term proportional to the velocity, $\dot{x}$, is the damping term that distinguishes the van der Pol oscillator from the simple harmonic oscillator, whose equation of motion is

$$\ddot{x} + \omega^2 x = 0. \tag{2}$$

The critical displacement a determines whether, at a given time, the system receives positive or negative damping. When the displacement $|x| > a$, the system receives positive damping, i.e., its effective kinetic energy decreases. But when the displacement $|x| < a$, the system receives negative damping, i.e., its effective kinetic energy increases. The result is that, when damping is gentle ($\eta \ll 1$), the system tends towards a periodic oscillation at its natural harmonic frequency, called a *limit cycle*. (In the jargon of dynamical systems theory, the limit cycle is an *attractor* of the system.)

Figure 3 depicts the phase portrait of this situation—a plot of the pairs of values $(x, \dot{x})$ constituting a solution to equation 1. The limit cycle depicted does not have an analytic solution, but it is quite close to $x(t) = 2\cos(\omega t)$, a solution to equation (2) for the simple harmonic oscillator with initial condition $(x_0, \dot{x}_0) = (2, 0)$. This allows one to describe

[10] The considerations apropos to the FQHE would also suffice for analogous discussions of emergent two-dimensionality in the simpler free electron-gas model and the integer quantum Hall effect from quantum models of the ordinary Hall effect, but discussing the more complex FQHE facilitates discussion with the views of Guay and Sartenaer (2016, 2018) in section 4.2.

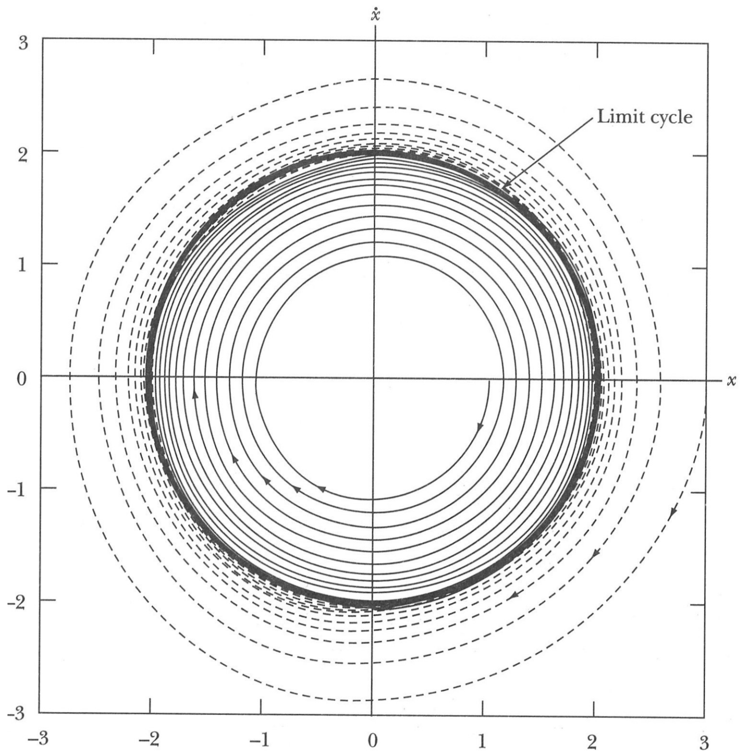


Figure 3: Phase diagram for some solutions to equation 1, with $a = \omega = 1$ and $\eta = 0.05$, from Thornton and Marion (2004, p. 153). The solid and dashed lines depict trajectories with respective initial values $(x_0, \dot{x}_0)$ of $(1, 0)$ and $(3, 0)$.

emergent *harmonic* periodicity in the van der Pol oscillator.[11]

It takes a few steps to show this. Let Φ denote the union of the phase space trajectories with endpoints and of any duration that are solutions to either the van der Pol or harmonic oscillator (equations 1 and 2, respectively). What structure on it is appropriate to describe how these trajectories are similar to one another? Suppose that one is interested in how different solutions approximate one another for bounded spans of time. Supposing further that approximation of position x and velocity $\dot{x}$ are equally important, one might select the Euclidean metric on the phase space as a distance function between points of that space at a time: letting for brevity in what follows $X_0 = (x_0, \dot{x}_0)$ and $X'_0 = (x'_0, \dot{x}'_0)$, $d(X_0, X'_0) = ((x_0 - x'_0)^2 + (\dot{x}_0 - \dot{x}'_0)^2)^{1/2}$. This in turn generates the uniform metric d_U between trajectories $X(t)$ and $X'(t)$ with a common temporal interval I as their domain: $d_U(X, X') = \sup_{t \in I} d(X(t), X'(t))$. The uniform metric assigns a distance between two phase space curves as the maximum of the Euclidean distances between the points in the image

[11] Larger values of the damping parameter η have more skewed phrase portraits, depending also on the initial conditions (Thornton and Marion, 2004, p. 154). For more technical details on the van der Pol oscillator, see Grasman (1987), Guckenheimer and Holmes (1983, Ch. 2.1), and Kanamaru (2007). For the history and prehistory of van der Pol's contribution, see Ginoux and Letellier (2012).

of the one at a time with the points in the image of the other at the same time. It induces a class of similarity relations $\sim_\epsilon = \{(X, X') \in \Phi \times \Phi : d_U(X, X') < \epsilon\}$ for $\epsilon \geq 0$, and one can select a similarity structure $\mathcal{S}_\delta = \{\sim_\epsilon : \epsilon \geq \delta\}$. The similarity structure $\mathcal{S}_\delta$ represents a context in which approximation of one curve by another via the uniform metric is relevant down to distances of δ; below that, they are not relevant. This sort of context arises naturally when one is modeling for practical applications, where measurements and control of phenomena only matter up to some finite precision.

Consider in addition a property assignment $\nu : \Phi \to [0, \infty]$ that assigns to each trajectory a non-negative number representing its period, that is, the time needed for the trajectory to recur if it were continued deterministically according to its respective equation of motion. (Fixed points have a period of 0, and non-recurring trajectories have a period of ∞.) Two values for the period, T and T', are similar just in case they are close numerically; this is formalized with relations of the form $\sim'_\epsilon = \{(T, T') \in [0, \infty] \times [0, \infty] : |T - T'| < \epsilon\}$ for $\epsilon \geq 0$. One can then collect these together into a similarity structure $\mathcal{V}_{\delta'} = \{\sim'_\epsilon : \epsilon \geq \delta'\}$ representing finite precision, as with the case of $\mathcal{S}_\delta$.

Now given the similarity spaces $(\Phi, \mathcal{S}_\delta)$ and $([0, \infty], \mathcal{V}_{\delta'})$, one can investigate questions about the emergence of finite periodicity in the van der Pol oscillator. For concreteness, consider the sort of phase space trajectories depicted in figure 3, denoting them by $A \subseteq \Phi$. These are non-recurring, so have period ∞. Let the periodic limit cycles, which for small η approximate simple harmonic motion (solutions of equation 2), be denoted by $B \subseteq \Phi$. These are recurring, so have finite, positive periods. For oscillators actually described by the van der Pol equation (1), the elements of A are more fundamental than those of B at least in virtue of their greater empirical adequacy. As long as one cannot measure phase space trajectories with infinite precision, i.e., $\delta, \delta' > 0$, the elements of both A and B will have overlapping domains of application, if this is understood as empirical adequacy: they will both be adequate to model some of the same phenomena *for sufficiently late times*. This is because the elements of B are attractors for the van der Pol dynamics: given δ, for sufficiently late-time segments of an element in A, each will be related to an element of B by a similarity relation in $\mathcal{S}_\delta$. Thus, this example meets the informal conditions of application for the emergence of finite periods.

What about the formal conditions of application? Without examining the similarity structures $\mathcal{S}_\delta, \mathcal{V}_{\delta'}$, one can already conclude that finite periods are weakly diachronically emergent in B with respect to A at sufficiently late times, simply because $\nu[A] = \{\infty\}$ while $\nu[B] = [0, \infty)$. What's more, $\mathrm{cl}_{\mathcal{V}_{\delta'}} \nu[A] = \{\infty\}$ no matter the value of δ', so finite periods are strongly diachronically emergent in B with respect to A at sufficiently late times. However, because the elements of B are limit cycles of trajectories in A, for any $\delta > 0$, at sufficiently

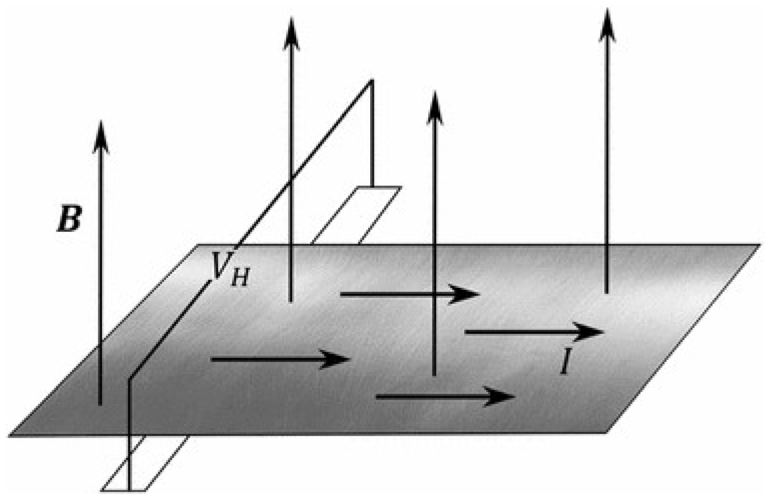


Figure 4: The shaded region represents the slab-like conductor or semiconductor, with I the current passing through it, B the magnetic field perpendicular to it, and V_H the Hall voltage measured transverse to the current.

late times trajectories in A become similar to elements in B according to the similarity relations in $\mathcal{S}_\delta$. Thus finite periods are not non-reductively diachronically emergent in B with respect to A at sufficiently late times, nor are they radically so (by modus tollens). Note however that if it were the case that $\delta = 0$, finite periods would be non-reductively (indeed, radically) diachronically emergent because $\mathcal{S}_0$ would include the identity relation, meaning that the trajectories of A would never become arbitrarily similar to those in B because they never merge. However, the overlapping domains of application of the models of these two sets reflects the finite precision of our measurement techniques, hence determines some $\delta > 0$.

3.2 The Fractional Quantum Hall Effect

Suppose an electric current I is run across an slab-like electrical conductor or semiconductor immersed in a magnetic field B perpendicular to it, as depicted in figure 4. A voltmeter attached to the sides of the conductor parallel to the current can detect a voltage difference V_H, the *Hall voltage*, proportional to the current I and the magnetic field B. (The constant of proportionality depends on the physical properties of the token (semi)conductor.) In analogy with Ohm's law, $V = IR$, one then defines the Hall resistance $R_H \propto B$. Classically, then, one expects and finds in most such materials a linear proportionality between the Hall resistance and the applied magnetic field.

For certain semiconductors, however, it was discovered that at sufficiently low temperatures, quantum effects arise in which the Hall resistance itself becomes *quantized*, yielding the quantum Hall effect (QHE):[12] $R_H = h/\nu e^2$, where h is Planck's constant, e is the electron charge, and ν is a number known as the *filling factor*. What values can the filling factor take on? If one assumes that the electrons in the material move freely and with negligible interactions—a so-called "free electron gas"—then one can show that the electron energies take on discrete values called *Landau levels* that depend on the magnetic field B. Roughly speaking, the higher the magnetic field, the higher the level of degeneracy to these levels, meaning the larger number of electrons that can occupy that energy state. When the occupancy of a Landau level is constant, the filling factor will take on a constant, integer value. Thus, as the magnetic field is increased, this leads to "plateaus" of Hall resistance as all or almost all electrons reside at or below the specified Landau level. This is depicted in the leftmost part of figure 5.[13]

However, just to the right of that part of the figure one will notice many *fractional* values for the filling factor. The discovery of these extra plateaus—the FQHE—by Tsui et al. (1982) challenged the explanation above. There is consensus that this novel behavior arises because, at sufficiently low temperatures, the assumption that the electrons are non-interacting is no longer effective: one must model the electrons not as free particles but with, roughly, a Coulomb potential governing their interaction. However, this is not the (beginning of the) end of the explanation, for the usual methods of perturbation and mean-field theory for treating interactions don't work in this case because of the high degree of energy-level degeneracy (Lancaster and Pexton, 2015, p. 348).

This has provided an opportunity for physicists to speculate more freely and argue for a new physical mechanism describing how these interacting electrons give rise to fractional filling factors. As Bain (2016) adroitly observes, there are at least four competing such mechanisms: Laughlin ground states, composite bosons, composite fermions, and topological order, yet despite this apparent variety it is not clear whether these accounts are really just notational variants of each other. What they all have in common is that they first admit a non-relativistic, quantum electrodynamical (QED) Lagrangian density for the electrons with Coulomb interactions, which can be expressed in a particular well-adapted coordinate

[12] The temperature at which this occurs depends on the nature of the material. Traditionally such temperatures have been at low, but more recently the QHE has been observed in room-temperature graphene (Novoselov et al., 2007).

[13] As one can observe in the caption of figure 5, the longitudinal resistance against the main current I also vanishes at these plateaus, and this is a part of the phenomena of the QHE that must be explained. So, a full scientific explanation would need to take into account the correlated vanishing longitudinal resistance, too. Fortunately, the accounts provided below largely do just this, so for simplicity of presentation, in this essay I will focus just on the explanation of the Hall resistance plateaus.

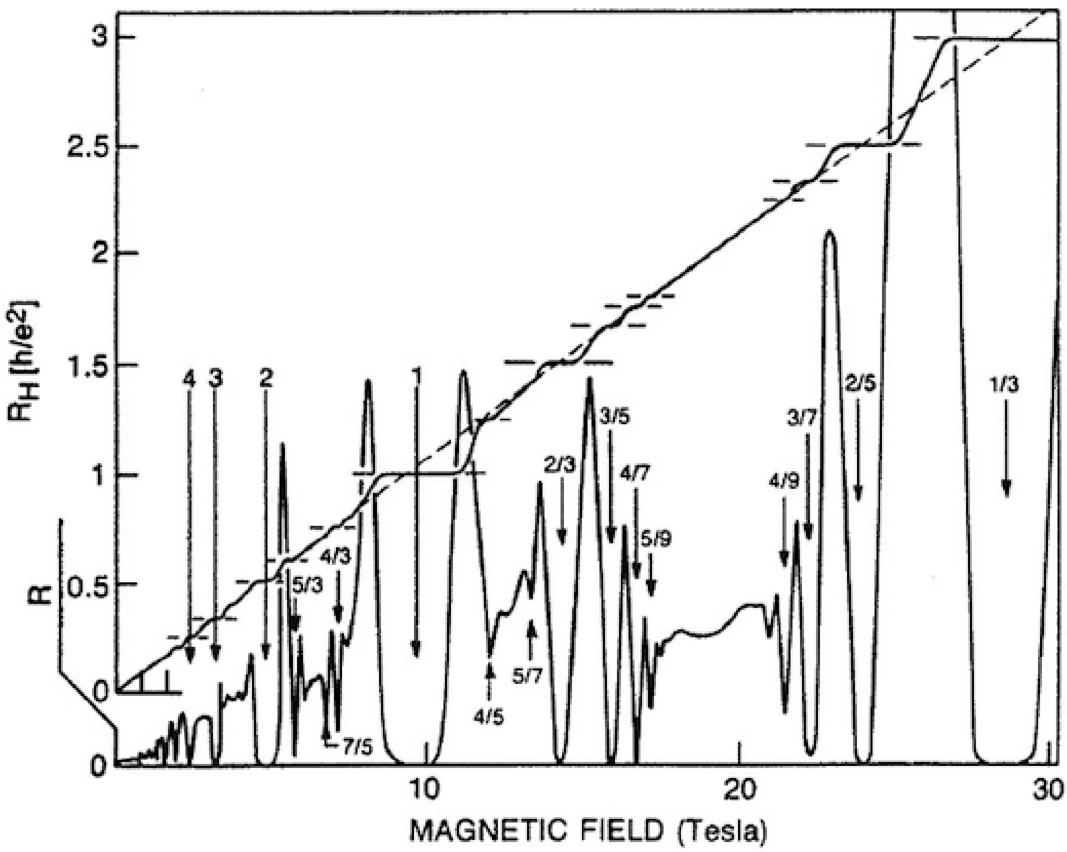


Figure 5: The upper plots depicts the Hall resistance, R_H, in units of h/e^2 plotted versus the external magnetic field in teslas; the dotted line represents the classical Hall resistance prediction, which is proportional to the magnetic field. Horizonal bars indicate the location and width of plateaus, while the vertical arrows indicate the filling factor, ν, ranging from 4 for low magnetic field strength to 1/3 for high. The lower plot depicts the longitudinal resistance, R, against the main current I as depicted in figure 4, versus the external magnetic field. (Stormer, 1992)

system as

$$\mathcal{L}[\psi, A_\mu] = \psi^\dagger(i\partial_0 - eA_0)\psi + \frac{1}{2m}\psi^\dagger(\partial_k + ieA_k)(\partial^k + ieA^k)\psi + V(\psi, \psi^\dagger), \tag{3}$$

where ψ is the second-quantized electron field, A_μ (for $\mu \in \{0, 1, 2, 3\}$) and A_k (for $k \in \{1, 2, 3\}$) are the electromagnetic (and magnetic) potentials (respectively), e and m are respectively the electron charge and mass, and V is the Coulomb potential. Instead of trying to solve this directly, one can take advantage of the fact that at low temperatures, high-energy modes of the electron field are suppressed, especially in the direction parallel to the external magnetic field. Integrating out these high energy modes and changing variables yields the effective Lagrangian density

$$\mathcal{L}_{\text{eff}}[a_\mu, A_\mu, j^\mu] = \frac{\epsilon^{\mu\lambda\rho}}{2\pi}(eA_\mu - (1/2\nu)a_\mu)\partial_\lambda a_\rho + j^\mu a_\mu, \tag{4}$$

where a_μ is an effective (Chern-Simons) field associated with the attachment of quantized electromagnetic flux to ψ, and j^μ is an effective quasiparticle field.[14] Here $\epsilon^{\mu\lambda\rho}$ is a *three*-dimensional antisymmetric symbol and all indeces run only from 0 to 2, as at low energies, excitations of any matter fields in the spatial dimension parallel to the external magnetic field are suppressed.

This quasiparticle field j^μ is unusual because it is *anyonic*. Bosonic fields ϕ_B are symmetric under permutation symmetry—$|\phi_B\phi'_B\rangle = |\phi'_B\phi_B\rangle$—and fermionic fields ϕ_F are anti-symmetric: $|\phi_F\phi'_F\rangle = -|\phi'_F\phi_F\rangle$. Anyonic fields ϕ_A pick up a phase factor θ other than 0 or π (modulo 2π) under permutation: $|\phi_A\phi'_A\rangle = e^{i\theta}|\phi'_A\phi_A\rangle$. The exact factor will depend on the field's other quantum numbers. Most accounts of anyons presuppose that they must reside only in two spatial dimensions.

Thus the effective Lagrangian, equation 4, is notable not just for the new effective fields by which it is characterized compared with equation 3, but also because it is a *two-dimensional* and *topological* QED Lagrangian. It is two-dimensional for the reasons just mentioned; it is topological because the Galilean temporal and spatial metrics do not appear. The second term of equation 3 tacitly depends on the spatial metric h^{ab} because it defines $\partial^k = h^{jk}\partial_j$ and $A^k = h^{jk}A_j$, while no index contractions in equation 4 evoke either the temporal or spatial metric, as the antisymmetric symbol $\epsilon^{\mu\lambda\rho}$ is (up to a choice of sign) uniquely determined regardless of the metric, and j^μ is defined from the beginning as a contravariant field.

With all of the foregoing details exhibited, one can ask: How does a FQHE state exhibit diachronic emergence? What are the relevant emergent properties? The answer depends

[14] For variations on how to derive $\mathcal{L}_{\text{eff}}$, see, e.g., Zee (1995, p. 110), Wen (2004, p. 298), Zee (2010, p. 326), Fradkin (2013, p. 502), and Lancaster and Blundell (2014, p. 419).

somewhat—as it should—on the chosen comparison class of models A, in the terminology of section 2. Consider in the first place the models which are variational solutions to the Euler-Lagrange equation for the QED Lagrangian, equation 3. These have a domain of application that includes the electrons in a (semi)conductor at a range of temperatures. Meanwhile, variational solutions to the Euler-Lagrange equation for the effective Lagrangian (equation 4)—the models B, in the terminology of section 2—have a domain of application that includes only sufficiently low-temperature electrons in a relatively thin slab-like semiconductor in a Hall experiment (i.e., as depicted in figure 4). So, if in a Hall experiment the temperature is gradually lowered, there is the potential for diachronic emergence of values of properties in the latter solutions once the domains of application of the two sets of solutions overlap.

Let $\nu_{\text{dim}} : A \cup B \to \mathbb{N}$ be the property assignment for the spatial dimension of a solution, and $\nu_\theta : A \cup B \to [0, 2\pi)$ be that for the exchange phase of the field considered in the models. Clearly $\nu_{\text{dim}}[A] = 3$ while $\nu_{\text{dim}}[B] = 2$; since ψ in equation 3 is an electron field and electons are fermions, $\nu_\theta[A] = \pi$, while $\nu_\theta[B] = \vartheta \neq \pi$. Thus two-dimensionality and anyonic fields are weakly diachronically emergent in the models of B with respect to the models in A.[15]

To determine whether these examples are also of strong emergence requires a similarity structure on their spaces of property values. I suggest that no dimension is arbitrarily similar to another except for itself; this entails that its similarity structure $\mathcal{V}_{\text{dim}}$ includes at least the identity relation. Even without other commitments to the elements of $\mathcal{V}_{\text{dim}}$, this entails that $\text{cl}_{\mathcal{V}_{\text{dim}}}[\nu_{\text{dim}}[S]] = \nu_{\text{dim}}[S]$ for any $S \subseteq A \cup B$, hence entails that two-dimensionality is also strongly diachronically emergent. As for the permutation phase factor, let $\sim_\epsilon^{\text{mod}} = \{(\alpha, \beta) \in [0, 2\pi) \times [0, 2\pi) : |\alpha - \beta| < \epsilon\}$ and $\mathcal{V}_\delta^{\text{mod}} = \{\sim_\epsilon^{\text{mod}} : \epsilon \geq \delta\}$. If $\delta < |\vartheta - \pi|$, then $\vartheta \notin \text{cl}_{\mathcal{V}_\delta^{\text{mod}}}[\nu_\theta[B]]$. In other words, if one can distinguish permutation phases with sufficient precision, one will not regard ϑ as being arbitrarily similar to π, and the anyonic fields are also strongly emergent.

To determine whether these examples are also of non-reductive (or radical) emergence requires a similarity structure on their spaces of models. Here I have less to say because part of the mathematical development of this subject is still unsettled. The first point to notice is that one needs to determine how QED models in two and three spatial dimensions are related. If one regards each class as being entirely separable with respect to the other—i.e., that no model of one is arbitrarily similar to a model of another—then the case for non-reductive emergence is strengthened. The issue here is not the derivation of $\mathcal{L}_{\text{eff}}$ from $\mathcal{L}$—the renormalization procedures used in that derivation are designed to maintain similarity of

[15] I have not included in this litany any property like "being topological" or "belonging to a topologically invariant theory" because these are relations between a model and a class of models characterized in a particular way (i.e., via the form of a Langrangian). Thus, they do not formally fall under the auspices of the sorts of property assignments I am considering in this essay as candidates for diachronic emergence.

relevant empirical features—but rather the initial assumption about the spatial dimension of the system. Indeed, most approaches to deriving the FQHE begin with a model of a quantum system in two dimensions. However, as Shech (2015) emphasizes, one cannot assume that just because it is convenient for theoretical derivation of the FQHE to assume from the outset that the quantum system resides in two spatial dimensions, it does not follow that a derivation is not possible from the more realistic assumption of a quantum system in three spatial dimensions. Shech (2015, p. 1086) conjectures that it should be possible to provide a limiting family of characterizations of the three-dimensional system, in the limit as the boundary conditions of one of the spatial dimensions squeeze the state to arbitrarily small support in that dimension. Another possibility would be to provide an account of the FQHE directly in three spatial dimensions, as Halperin (1987, p. 1915) has suggested is possible with generalizations of the Laughlin states. I will return to this question in my discussion of Guay and Sartenaer (2016, 2018) in section 4.2.

4 Comparisons

Although there are many philosophical accounts of emergence of varying detail and development, relatively few both concern diachronic (as opposed to synchronic) emergence and aim at capturing or explicating the use of emergence concepts in science (as opposed to in metaphysics or philosophy of mind). But, there are some. So, in this section, I compare the similarity structure approach to diachronic emergence with two others to which I've found it to be closely related: in section 4.1, that of Rueger (2000) and McGivern and Rueger (2010), which they sometimes call "physical" emergence; and in section 4.2, that of Humphreys (2016b) and Guay and Sartenaer (2016, 2018), dubbed "transformational" emergence. In each case I'll argue that the similarity structure approach should be preferred on grounds of generality and of capturing the details of the foregoing scientific examples.

4.1 Physical Emergence

Rueger (2000) and McGivern and Rueger (2010) (hereafter RMR) propose an account of diachronic emergence that bears some similarities with the present account. I'll describe these similarities before enumerating the differences between the two and why the present account should be preferred: it both better captures essential details of examples of diachronic emergence and has broader scope. To illustrate the first advantage I'll focus on the example of emergent finite periodicity in the van der Pol oscillator, an example RMR also examine in detail.

RMR also focus on emergent properties[16] and agree that they are contrastively novel (McGivern and Rueger, 2010, p. 215)—that is, they obtain for some system and not for another, relevant contrast system. They also agree that "the requirements for emergent properties have natural and fairly precise counterparts in . . . physics and that a unified account of these requirements becomes possible" (Rueger, 2000, p. 299) by abstracting from them. Moreover, "an account of emergence should be able to accommodate both" synchronic and diachronic types (McGivern and Rueger, 2010, p. 215) and "Answering [whether a putative case of emergence is ontological] will require some further assumptions about what features of theories are ontologically significant, and when" (McGivern and Rueger, 2010, p. 214).

One relatively minor point of disagreement concerns the "non-reducibility" of the emergent property to properties of the system in the comparison class. RMR insist on this, and indeed emergence is often seen as antithetical to reduction. When they seek to define non-reducibility, though, they relate it inextricably to novelty: "we see the two criteria as two sides of the same coin: emergent phenomena are typically taken to be not only novel but in some way 'qualitatively' novel, and talk of irreducibility often seems intended to capture just this distinctive feature" (McGivern and Rueger, 2010, p. 215). What they have in mind is some failure of a "smooth" limit-type reduction (Nickles, 1973) between the models with the putative emergent property and those of the comparison class. This is a minor contrast with the similarity structure approach because it can be shown that (perhaps unsurprisingly) only non-reductive and radical emergence are associated with a failure of limiting-type reduction, while weak and strong emergence are in fact compatible with it (Fletcher, 2020, ming).

A more significant point of disagreement lies in what exactly the comparison class is supposed to be. For RMR, this involves comparison of the same system's coarse-grained properties with its fine-grained properties in the case of synchronic emergence.

> For *diachronic* emergence, however, the relevant distinction is between properties of a system at one time and those at a later time. . . . [In particular,] the behavior of the system at a time is emergent with respect to the system at an earlier time if some parameter in the base has changed its value slightly during the time interval and the later behaviour is 'novel' compared to the behaviour of the old system and irreducible to it. (Rueger, 2000, p. 300)

Within the depiction of diachronic emergence in figure 2, this corresponds to the elimination

[16] Rueger (2000) speaks of emergent properties, while McGivern and Rueger (2010, p. 215) write that "We take it that emergence must involve emergent *behavior* of some sort . . . . However, since it is also common to speak of emergent properties, at times we will do this as well. In these cases, such properties can be understood in the sense of being the property of having a particular sort of behavior." However, they don't define what a behavior is beyond having a particular temporally extended property.

of the class B (hence consideration of its domain of application, D_B). Instead the comparison class is the set of models in A whose domain of application are before some designated time, with the models to which the putative emergent properties pertain as those whose domain of application is after that time.

To illustrate this, RMR also use the case of the van der Pol oscillator considered in section 3.1.[17] However, if they do not consider the periodic trajectories in the class B, how do they determine what the novel, emergent properties are? Here, "novelty of behavior is to be characterized in terms of topological differences between the representations of a system's behavior before and after a control parameter reaches or crosses a critical value" (McGivern and Rueger, 2010, p. 219), where the control parameter in question is the damping coefficient, η. In this case, "the novel property of the van der Pol oscillator with $\eta > 0$ considered above would be 'having a limit cycle', a feature missing from the oscillator with $\eta = 0$" (Rueger, 2000, p. 303). The "topological differences" refer to the fact that, e.g., the solutions to equation 1, the van der Pol equation, depicted in figure 3 cannot be smoothly deformed in phase space to the solutions of equation 2, the equation for the simple harmonic oscillator.

As the reader may have noted, however, η is not a temporal parameter in the van der Pol equation, so two van der Pol oscillators with different values thereof are simply not diachronically related, hence not an example of diachronic emergence. If one constructs a *different* system, making η a function of time, then that system is not a van der Pol oscillator because the characterization of a van der Pol oscillator requires η to be constant. But properties of the solutions to the van der Pol oscillator equation (1), such as having limit cycles, cannot be derived if η is some arbitrary continuous function of time. In particular, having limit cycles is a property of van der Pol systems in the infinite-time limit. It's therefore probably most charitable to re-insert the reference class B for diachonic emergence, as depicted in figure 2, representing solutions to the van der Pol and harmonic oscillator equation, with the models in A being solutions to an analogous equation allowing η to vary in time, starting with a constant $\eta > 0$, then transitioning to a constant $\eta = 0$.[18]

This concession would bring RMR's account of physical emergence closer to that of the similarity structure approach. But its criterion of novelty is too specific to dynamical systems theory to accomplish their goal of a unified account of emergence: even in physics, not all interesting cases of diachronic emergence involve models that are trajectories in a classical

[17] Technically, while Rueger (2000) considers this case, McGivern and Rueger (2010) consider the simpler example of a damped harmonic oscillator, but ostensibly just for ease of presentation: they maintain that the van der Pol oscillator illustrates the same relevant features is a more realistic way (McGivern and Rueger, 2010, p. 221).

[18] Guay and Sartenaer (2016, p. 308n10) note similar interpretational difficulties for Rueger (2000) but retain the "same system, different times" interpretation; they do not consider the technical difficulties it provides for this example as one of diachronic emergence.

phase space, as the FQHE from section 3.2 illustrates. Moreover, even within dynamical systems theory, "topological differences" between the trajectories in phase space is at best a sufficient but not necessary condition for diachronic emergence. As I mentioned before, and as depicted in figure 6, for large values of the damping parameter η, the limit cycle of the van der Pol oscillator is no longer well-approximated by the phase space trajectory of a simple harmonic oscillator, which is a circle (or more generally, an ellipse). It becomes instead jerky, alternating between relatively slow-motion and fast-motion phases. Even Rueger (2000, p. 311) considers this to be "a very different type of behavior," and one could construct a process similar to the one described above—varying the damping parameter from $\eta = 0$ to $\eta \gg 1$—to provide an example of diachronically emergent jerkiness. Yet the trajectories are all closed loops, with no topological differences between them. Instead of specifying the criterion for emergent novelty once and for all, the similarity structure approach allows contextual information, such as what and how models represent, what can be measured, and how precisely, to dictate the relevant ways for different scientific models of phenomena to be similar. This similarity structure then determines different grades of novelty. Thus even though diachronic emergence has many exemplars in dynamical systems, the similarity structure approach applies also to models beyond.

4.2 Transformational Emergence

In devising and propounding *transformational emergence* (TE), Humphreys (2016b) and Guay and Sartenaer (2016, 2018)—hereafter, GS—have been motivated to develop an account of emergence that is useful to science and instantiated in the world, while being sensitive to the philosophical issues involving the intersection of emergent phenomena with causal powers, all without invoking too much abstruse metaphysics. In this section, I'll reconstruct TE first to draw out its similarities and differences with the similarity structure approach. Then, I will re-examine GS's account of the FQHE as an example of TE to show that TE does not adequately capture the physics of and empirical evidence about the FQHE. That said, while GS seem to have more imperialistic ambitions for TE, Humphreys (2016b, p. 56) allows that it may be only one type of emergence among many. Thus, I'll conclude with a sketch of how, with some modification, TE can be modeled within the similarity structure approach, illustrating this with a variation on the FQHE example.

To begin with its definition, Humphreys (2016b, p. 60) writes that

> Transformational emergence occurs when an individual [or property or state] a that is considered to be a fundamental element of a domain D transforms into a different kind of individual [or property or state] a*, often but not always as a

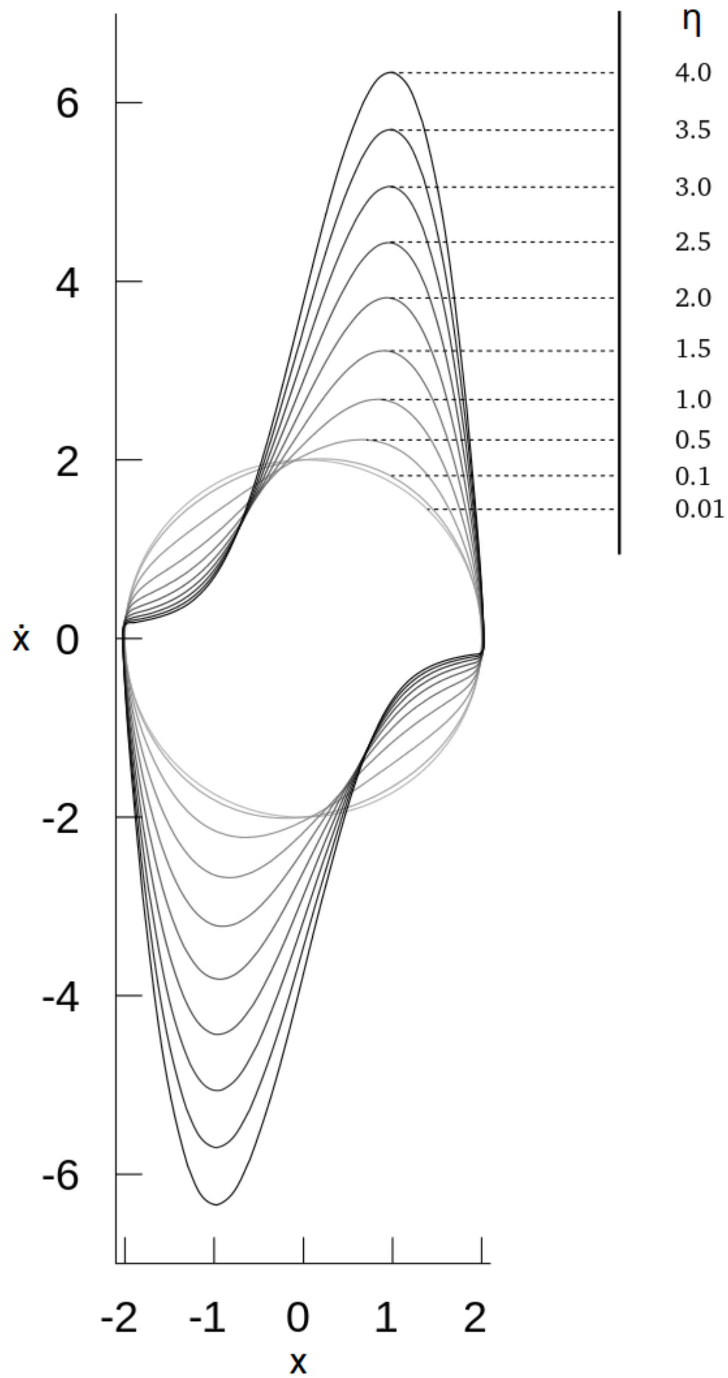


Figure 6: Limit cycles of the van der Pol oscillator (equation 1) for $a = \omega = 1$ and a variety of indicated values of the damping parameter η. Adapted from an image by Widdma, CC BY-SA 3.0, `https://commons.wikimedia.org/w/index.php?curid=9391038`.

result of interactions with other elements of D, and thereby becomes a member of a different domain D*. Members of D* are of a different type from members of D. They possess at least one novel property and are subject to different laws that apply to members of D* but not to members of D.

Here, a property is novel with respect to the domain D when it "is not included in the closure of D under closure criteria C that are appropriate for D" (Humphreys, 2016b, p. 62). In a word, "transformational emergence can be produced when at least one essential property of a fundamental entity is changed, with an accompanying change in domains" (Humphreys, 2016b, p. 70).[19] (Here, a "domain" functions much like a theory or collection of models in the similarity structure approach.)

GS give an essentially equivalent definition, although they emphasize three further points. First, they focus on states rather than individuals or properties, and second, they emphasize that the process of transformation must be spatiotemporally continuous, but these two don't materially distinguish their account from Humphreys'. Third, they require not just that a* (as described in the quotation above) have novel properties and belong to a different domain D*, but that at least one of the novel properties is forbidden according to the laws of D, hence the laws of D and D* cannot be simultaneously consistent (2016, p. 303). For GS, emergence "is about impossible phenomena (according to the pre-emergence laws) that, upon emergence, become possible and even actual (due to the advent of new laws that reconfigure the space of possibilities)" (2018, p. 225).

TE accords with the similarity structure approach in its focus on contrastive novelty as essential for emergence, even with GS's modal emphasis: the image of a property assignment on a collection of models yields the values for that property that are possible according to those models, so there is certainly a sense in which even values of properties weakly emergent in a collection are impossible in their comparison class. Even though TE does not focus exclusively on properties as the sorts of things that can be emergent, the further allowances for states and individuals are only cosmetic differences when the focus is on scientific models, as the presence and absence of certain states and individuals is easily encoded into a property assignment on models. TE does, however, take from the beginning an ontological line on emergence, so would perhaps see a larger gap between individuals, properties, and states than the similarity structure approach: even though these can all be represented in much the same ways in scientific models, this does not necessarily mean that they are themselves the same sort of thing. Of course, the problem with this minimal ontological commitment is

[19] Earlier, Humphreys (2016b, p. 61) begins to append a subscript "S" to "fundamental" to denote a "synchronic" notion of fundamentality denoting indivisibility and immutability. Since I do not intend to contrast this notion of fundamentality with any other, I have omitted this subscript in the quotation for clarity of expression.

that it makes it harder to interpret scientists' claims about emergence when they use theories to which they seem not ontologically committed.

A potentially more significance difference, however, concerns the sort of novelty at issue in TE. As Shech (2019, §6) observes, one could read its "novelty" criterion—following them, call it NOV—in a strong and a weak way:

$\mathbf{NOV_{strong}}$: "there is some *in principle* argument ... to the effect that [B] is not derivable from [A]" (Shech, 2019, p. 605);

$\mathbf{NOV_{weak}}$: there is "an in principle derivation of [B] from [A], but ... the laws associated with each are substantially different so as to justify the idea that [B] is novel with respect [to A]" (Shech, 2019, p. 605).[20]

NOV_{strong} closely resembles non-reductive or radical emergence, while NOV_{weak} closely resembles weak or strong emergence, in the framework of the similarity structure approach to emergence. On the one hand, Humphreys (2016b, p. 8) seems to allow both sorts of novelty: "The traditional contrast between emergence and reduction, with its emphasis on synchronic reductions, fails to capture an important class of diachronically emergent phenomena, and we should not insist that examples of emergence are always a result of failures of synchronic reduction." On the other, GS seem to insist on the strong version, NOV_{strong}. When it comes to the diachronic emergence of a state S_2 of a dynamical system from a state S_1, they "operationalize" NOV_{strong} as the requirement that "S_2's dynamics as described by [B] is not continuously deformable into S_1's dynamics as described by [A]" (2016, p. 305).[21] This notion of "continuous deformation" is essentially the same sort of limit-type reduction (Nickles, 1973) between models that RMR invoked in their account of emergence in section 4.1. One might then suspect that its application runs into similar problems as RMR's.

As I alluded at the end of section 3.2, the mathematics are less well-developed in the case of the FQHE than in the case of the van der Pol oscillator, but there is still a definite outline to the state of the field. To see this, I'll focus on how GS argue for NOV_{strong} in the case of the states representing the FQHE. First, they divide the models of QED into two relevant sets: those with three spatial dimensions, QED_{3+1}, and those with two, QED_{2+1}. They write that "it should not be possible to obtain QED_{2+1} from QED_{3+1}, and moreover, there is no continuous limit that could get QED_{2+1} from QED_{3+1}, to the effect that one should not consider QED_{2+1} as just being QED_{3+1} with one less dimension" (2016, p. 315). The reason for why "it is not possible to continuously deform one model into the other [is]

[20]For uniformity of notation I have replaced Shech's M_1 and M_2 with A and B, respectively.

[21]See also Guay and Sartenaer (2018, p. 227). As with the quotation from Shech above, for uniformity of notation I have replaced GS's M_1 and M_2 with A and B, respectively.

because there is no analog to the Chern-Simons term in four dimensions" (2018, p. 229). The Chern-Simons term is that of equation 4 containing the filling factor ν, and for GS it suffices to make the QED_{2+1} states representing the FQHE novel with respect to any QED_{3+1} state.[22] GS (2018, p. 229n17) support this with a Nobel Lecture quotation from one of the original FQHE theoreticians, Robert B. Laughlin (1999, p. 869): "The fractional quantum Hall state is *not* adiabatically deformable to any noninteracting electron state."

There are two principal problems with their assertion of this impossibility. First, as Shech (2019, p. 608) has already noted and as Bain (2016) discusses in some detail, one does in fact derive the effective Lagrangian $\mathcal{L}_{\text{eff}}$ from the standard non-relativistic QED Lagrangian $\mathcal{L}$ through the process of renormalization. The existence of a Chern-Simons term is not an impediment to this derivation. This is moreover not in conflict with the quotation from Laughlin, as $\mathcal{L}$ represents *interacting* electron states. Laughlin is rather emphasizing what I described in section 3.2, that one cannot derive the FQHE states using perturbation theory on states representing a *free* electron gas. If there *is* an impediment, it is rather the dimensional transition from QED_{3+1} to QED_{2+1}, as I remarked at the end of section 3.2.

However, there is still reasonable hope for this because of the second problem: Tang et al. (2019) has recently observed the FQHE—in particular, filling factor $\nu = 1/3$—in a material that cannot be adequately modeled as being two-dimensional. They reference theoretical work suggesting that one approach to the FQHE, Laughlin states, can be generalized from two to three spatial dimensions (Halperin, 1987). This shows that FQHE states are in principle describable within QED_{3+1}, although there is still much theoretical work to be done to flesh them out.[23] Importantly, this possibility is compatible with the novel two-dimensionality and anyonic statistics of states adequately modeled by the usual FQHE states; it only undercuts reasons to believe that these states cannot be derived, using appropriate approximations and idealizations, from more fundamental models of QED_{3+1}. Thus, it is still compatible with these novel properties' weak and strong diachronic emergence within the similarity structure approach.

Does this mean that TE adequately captures the novelty of the states (or properties thereof) describing the FQHE, when novelty is interpreted only as NOV_{weak}? Not quite.

[22] In their earlier work, GS assert that another sufficient property for the FQHE states to be novel is that they exhibit anyonic ("fractional") statistics: "As far as we know, this possibility *does not exist* in 3+1 dimensions" (2016, p. 315). However, in their later work they implicitly retract this claim: "fractional statistics could in principle exist in a four-dimensional world. The real novelty is access to new topological quantum states" (2018, p. 228n16). As is implied in section 4, such statistics are certainly novel with respect to those in the QED_{3+1} treatment of electrons, but perhaps not with respect to other sorts of fields. An analogous issue with the FQHE states themselves will arise below in light of recent experimental work (Tang et al., 2019).

[23] Cf. GS's insistence that "A planar space seems necessary to be able to describe the FQH effect, and [this] is not the case for a very thin conductor" (2016, p. 317).

The reason concerns the demand that a fundamental element of QED_{3+1} *transform* into a different one of QED_{2+1}, with at least one distinct essential property. The diachronic process of exhibiting the FQHE takes a Hall experiment and gradually lowers its temperature to suppress high-energy electron modes. In this process, the QED_{3+1} electrons remain QED_{3+1} electrons, as their particular kinetic energies (which determine the temperature of the many-electron system) are not essential properties thereof. Pace GS's suggestion (2016, p. 318), if QED_{3+1} electrons were to gain new essential properties and thereby transform into a different type of particle in a Hall experiment, they would grossly violate physical law, as QED regardless of dimension forbids such a process through merely lowering the electron temperature.[24] The similarity structure approach to diachronic emergence precludes such transformations because the domain of application of the pre-emergence models extends into the post-emergence period—see the depiction of D_A in figure 2b.

Now, the reason that the advocates of TE posit this transformation of essential properties is to avoid the untoward consequences of so-called causal exclusion arguments. Here's how Humphreys (2016b, p. 71) describes one version:[25]

1. Every physical event E that is caused has a prior sufficient fundamental physical cause C.
2. If an event E has a prior sufficient cause C, then no event C* distinct from C that is not part of the causal chain or process from C to E is causally relevant to E.
3. The realm of the fundamentally physical is causally closed, and so all events in the causal chain or process from C to E are fundamental physical events.
4. No emergent event C* is identical with any fundamental physical event.

Therefore,

5. No emergent event C* is causally relevant to E.

Therefore,

[24] The same goes for so-called *fusion* emergence (Humphreys, 1997), in which the pre-emergence individuals no longer exist post-emergence. (Humphreys (2016b, Ch. 2.3) now sees fusion emergence only as a special case of TE.) See Lancaster and Pexton (2015, §6.2) for further criticisms of applying fusion emergence to the FQHE; they suggest instead a modification: "Rather than the fusion relation using up basal property instances, it instead *turns non-relational properties into inherently relational properties*" (2015, p. 355). This is clearly an example of TE, since essential properties of electrons are changed, and falls under the same criticisms as TE just discussed. However, since they have in mind a pre-emergence state modeled by *non*interacting electrons, it might fall under the case of *indirect* diachronic emergence, as described in the sequel.

[25] For simplicity of expression in this quotation I have omitted footnotes and subscripts "S" on "fundamental"—see footnote 19.

(a) Direct diachronic emergence (b) Indirect diachronic emergence

Figure 7: Diagrammatic comparison of direct and indirect diachronic emergence according to the similarity structure approach. Direct diachronic emergence requires that the domain of application of A, D_A, be wider temporally than the domain of application of B, D_B. This permits emergent property values in B with respect to A. Indirect diachronic emergence introduces a new comparison class, C, whose domain of application D_C overlaps with D_A; the later "mediates" the emergence of property values in B with respect to C.

6. Emergent events are excluded from causally affecting fundamental physical events and hence are causally dispensable.

TE (and fusion emergence, as described in footnote 24,) rejects premise 3 or 4 because the emergent C* is a member of a different domain, D*, which is either not a fundamental physical domain, or is one, respectively. By contrast, the similiarity structure approach to emergence is agnostic on these points. One option for an advocate thereof is to accept the argument, providing reason to deny the ontological nature of the types of emergence described. Another is to reject premise 2, the "no causal over-determination" thesis, as not resting on any coherent or plausible theory of causation (Sider, 2003; Hitchcock, 2012; Roche, 2014). I prefer this latter option, but I shall not argue for it further here.

All that said, there is room for accommodating a version of TE applied to FQHE within a slight extension to the similarity structure approach to emergence. This extension focuses on the *non*-interacting ("free") electron gas models as the comparison class instead of the general QED models that are solutions to the Euler-Lagrange equations with Lagrangian $\mathcal{L}$ given by equation 3. Call the former models C and the latter models (as before) A, with the FQHE models denoted by B. Properties such as non-interaction are synchronically emergent in C with respect to the models in A, and the properties of models in B are certainly different from these, but they have no overlapping domain of application with those in C: the non-interaction models are adequate in a Hall experiment only when the temperature is sufficiently high, while the interaction models are adequate only when the temperature is sufficiently low. The models don't fulfill the informal conditions of application for the similarity structure approach to diachronic emergence.

Nevertheless, there is a sense in which these models' domains of application are connected through that of A. Call the usual situation *direct* diachronic emergence, and this new situation *indirect* diachronic emergence. They are depicted in figure 7. In direct diachronic emergence, a property value of B is novel with respect to the property values in A in one of the partially ordered respects described in section 2. In indirect diachronic emergence, the comparison class C has no overlapping domain of application with B, but they share a domain of application with A, whose models are at least as fundamental as those in B and C. Moreover, models of C are adequate only in the "before" period, and models of B are adequate only in the "after" period. The allows for the "transformation" or "loss" of properties going from C to B that is essential to TE, mediated by A to determine the before/after distinction and ensure that the models are being applied to the same subject matter.[26] In the case of the FQHE, indirect diachronic emergence captures the idea that in the temporal process of a Hall experiment, "Certain individual properties of the electrons are lost to a collective set of properties of the FQH state. Electrons are still present in the system but, with respect to salient properties for the FQHE, they cannot be treated separately [i.e., as free particles] due to long-range entanglements" (Lancaster and Pexton, 2015, p. 356) or other interaction effects.

5 Beyond Time

The similarity structure approach to diachronic emergence better accommodates the scientific details of examples such as the van der Pol oscillator and FQHE than those of RMR's physical emergence and the TE of Humphreys and GS, respectively. Its extension to "indirect" diachronic emergence accommodates cases of "transformation" that TE attempts to explicate. Moreover, it offers finer-grained distinctions about the sort of novelty essentially involved in emergent properties, with some versions (weak and strong) compatible with reduction and others (non-reductive and radical) not compatible. Its account of diachronic emergence is unified with its account of synchronic emergence, in the sense that the criteria for comparative novelty and relative fundamentality are the same; the only difference consists in how the domains of application of the models in question overlap.

Indeed, the role of time in diachronic emergence is merely to delineate the domain of application of the models in which the putatively emergent property value resides from the

[26] Humphreys (2016b, p. 60) considers this possibility and notes that, on his account, when A (what he calls a "domain") is disjoint from B and C, it will not count as ontological emergence. By contrast, according to the similarity structure approach, whether it counts as ontological will depend, as always, on the realist credentials of the (relevant parts of the) models. This is a sense in which indirect diachronic emergence only accommodates a version of TE.

broader domain of the models of the comparison class. Other, non-temporal means of doing so—spatial location, temperature, energy, etc.—would produce a structurally analogous version of the synchronic/diachronic distinction applied to emergent properties. Thus, the similarity structure account of emergence reveals that this temporal distinction is not so fundamental to the categorization of types of emergence as has been thought. What matters is rather any criterion for delineating domains of application.

This conclusion equips the similarity structure approach well to extend to cases of emergence where the synchronic/diachronic distinction may not even be conceptually well-defined: the possible emergence of spacetime itself from theories of quantum gravity. These theories, albeit all still in development, posit a more fundamental structure to the world than that suggested by quantum theory or general relativity. Many (if not most) of these generically suggest this structure is non-spatiotemporal in some sense, including approaches such as canonical quantum gravity (Butterfield and Isham, 1999), causal set theory and other discrete structure approaches (Wüthrich, 2019; Crowther, 2016, Ch. 6), effective field theories (Crowther, 2016, Ch. 5), loop quantum gravity (Wüthrich, 2019; Huggett and Wüthrich, 2018; Crowther, 2016, Ch. 7), and string theory (Huggett and Wüthrich, 2018).

For instance, string and loop quantum cosmology describe a "pre-geometric" phase to the early universe where there are not yet spatiotemporal properties, which eventually "transitions" to a geometric, spatiotemporal phase, replacing the singularity of the Big Bang from non-quantum cosmological models of general relativity. But are not the very concepts of "before" and "transition" temporal concepts that could have no truck with non-temporal structures (Huggett and Wüthrich, 2018, pp. 1201–2)? One can instead set within the non-spatiotemporal quantum gravity models a different, endogenous scale—perhaps energy, entanglement, or something else—that functions to delineate different domains of application of these models. In some more restricted domain according to that scale, certain spatiotemporal models will be representationally adequate. Because the properties of temporality (and spatiality) are novel in those models, they will be emergent, too. It is up to researchers in quantum gravity (perhaps with a little help from their philosophy friends) to make these ideas precise in concrete programs for quantum gravity, but the similarity structure approach to emergence provides precise tools for drawing fine-grained conclusions about what property values are emergent, and in what sense, without assuming temporality. Approaches to emergence that make it essentially diachronic or that presuppose a strict synchronic/diachronic division will be of no obvious help.

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