# Similarity Structure and Emergent Properties

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**Abstract**

The concept of emergence is commonly invoked in modern physics but rarely defined. Building on recent influential work by Butterfield (2011a,b), I provide precise definitions of emergence concepts as they pertain to properties represented in models, applying them to some basic examples from spacetime and thermostatistical physics. The chief formal innovation I employ, similarity structure, consists in a structured set of similarity relations among those models under analysis—and their properties—and is a generalization of topological structure. Although motivated from physics, this similarity-structure-based account of emergence applies to any science that represents its possibilia with (mathematical) models.

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# 1 Introduction

The emergence of one or more properties of a system or state of affairs is commonly invoked in the physical sciences but rarely defined. Butterfield (2011a,b, 2014) goes some way to illustrate this usage (and its relation with concepts of reduction and supervenience) in various examples from physics, but, following tradition, does not provide a completely explicit or precise definition of emergence. The closest he comes to doing so is the statement that emergent properties are *novel* properties, where " 'novel' means something like: 'not definable from the comparison class', and maybe 'showing features (maybe striking ones) absent from the comparison class' " (Butterfield 2011a, 921).[1] But as Teller (1992) has protested, mere invocations of some sort of novelty or other are too vague and weak to be of much help. Thus the adjudication of Butterfield's and others' examples has continued to rely on informal intuitions and ostensive comparisons with purportedly intersubjectively agreed upon cases—hard to come by in the emergence literature.

Nevertheless, I believe Butterfield has set us in the right direction towards a precise and fruitful account of emergence for properties as described within scientific theories. My goal in the sequel is to take the next few steps. First, I follow Butterfield in focusing on emergence as it applies to scientific models, e.g., of theories, rather than of systems or objects directly. I then propose understanding emergence as *comparative dissimilarity*: a

[1]Butterfield (2011b, 1066) also demands that emergent properties be "robust," but this requirement is later dropped (Butterfield 2014). I agree with this more minimal demand, but I do take robustness to be indicative of an emergent property being *interesting*—see section 3.

property, as represented in a model, is emergent with respect to a comparison class just when is it relevantly dissimilar to the properties of the members of that class. This dissimilarity can come in different sorts, so in section 3 I distinguish four different types of emergence—weak, strong, non-reductive, and radical—partially ordered in strength. I also suggest several features that make examples of emergent properties more *interesting* than others.

The chief formal innovation I employ, introduced beforehand in section 2, is *similarity structure*, a structured set of similarity relations among those models under analysis—and perhaps their properties. Similarity structure is a generalization of topological structure.[2] After explaining the interpretation and significance of these structures and using them to define in section 3 the aforementioned types of emergence, I apply the definitions in section 4 to a couple of examples from spacetime and thermostatistical physics. Although these applications are both from physics, my definitions could extend to any sciences that use (mathematical) models to represent phenomena.

In section 5 I discuss some possible cases for such extension as well as three notable features of this account in general. First, some of its types are compatible with reduction, while others are not. This helps understand conflicting intuitions regarding this compatibility. Second, my account of emergence is relational and contextual. It is relational in that emergence is defined only relative to a comparison class, in agreement with Butterfield; it is contextual in that this comparison class, as well as the relevant notions of similarity, may vary according to the context of investigation. They are not

[2]It is more directly a generalization of uniform structure (Willard 1970, Ch. 9), but this genealogy will not make a significant difference in what follows.

determined from the formal structure of the models themselves. Third, the degree of precision with which one can identify which properties are emergent is limited in principle only by the precision with which one can specify the models of interest and the relevant ways those models are similar.

Before continuing in more detail, a few qualifications are in order in light of the apt remark of O'Connor and Wong (2015) that "Emergence is a notorious philosophical term of art," one with a variety of conflicting usages and definitions. I have already specified that the notion of emergence with which I am concerned here pertains to properties as described in scientific or formal models (rather than, say, to entities or substances). Following the distinctions urged by Humphreys (2016), this sort of emergence is certainly epistemological, as I describe in section 3, in the sense that it is associated with a failure of deduction, of a sort, of the emergent property; sometimes it will count as conceptual emergence too when the emergent properties involve novel concepts. Whether a particular example is also a case of ontological emergence will depend on whether one's (contextually appropriate) attitude toward the models in question (and in particular the putatively emergent property thereof) is more realist or anti-realist, but that question can be set aside for present purposes, as its answer does not substantively affect the formal features of the analysis. Further, the notion of emergence here is synchronic: it will describe the relationships of models and novel properties thereof, not how properties of systems or states of affairs arise in time.[3]

[3]That said, I do think the present account could be applied to cases of diachronic emergence: roughly, a model with an emergent property with respect to some comparison class becomes empirically viable for a particular system after some time, while a model from the comparison class remains viable throughout. But further elaboration is beyond

# 2 Similarity Structure

In order to define emergence as relative dissimilarity, first one must define similarity structure, whose constitutive unit is the qualitative, binary similarity relation.

**Definition 1.** A *similarity relation* $\sim$ on a set $X$ is a non-empty binary relation on $X$ that is *quasi-reflexive*: for all $x, y \in X$, if $y \sim x$, then $y \sim y$ and $x \sim x$.

The relation $y \sim x$ is interpreted as "$y$ is similar to $x$". Quasi-reflexivity requires that if any element is similar to another, then both the former and the latter are similar to themselves. Previous authors (Carnap 1967; Schreider 1975; Mormann 1996; Konikowska 1997) have commonly demanded a similarity relation to be not just quasi-reflexive, but *reflexive*—$x \sim x$ for every $x \in X$—and *symmetric*: if $y \sim x$ then $x \sim y$. However, while many useful similarity relations do satisfy these conditions, there are also some that do not. For example, one can consider the relation of (weak) observational indistinguishability among relativistic spacetimes (Malament 1977; Manchak 2009) as a type of non-symmetric similarity relation: one spacetime $(M, g)$ is similar in this sense to another, $(M', g')$, just when the past of each event in $(M, g)$ is identical with the past of an event in $(M', g')$. Each observer in $(M, g)$ cannot tell whether they are in $(M, g)$ or $(M', g')$, but some observers in the latter can tell that they are not in $(M, g)$. Regarding reflexivity, a mundane example suffices: among all people, one can consider two to be similar in party affiliation just when they are members of the same political party. Anyone registered with a political party is similar in party affiliation to themselves, but one who is unregistered is not, because they have *no* such affiliation.

my present scope.

In many contexts there are a variety of relevant ways in which the elements of a collection can be similar to one another. Considering multiple similarity relations on $X$ allows comparisons of similarity in multiple respects.

**Definition 2.** A *similarity space* is an ordered pair $(X, \mathcal{S})$, where $X$ is a set and $\mathcal{S}$ is a non-empty set of similarity relations on $X$, called a *similarity structure*.

This is fruitful precisely when there are multiple respects that could be relevant. For example, if there is a quantitative property relevant for comparison, then one could have a whole class of similarity relations, one for each positive $\epsilon$, relating the models whose quantitative property values are within $\epsilon$. (I discuss properties and their assignments more at the end of this section.)

Of particular interest will be the elements in a similarity space that are *arbitrarily* similar to others, that is, similar in each of the respects that a similarity structure represents. To define these, let $\wp X$ be the power set of $X$, i.e., the set of all subsets of $X$.

**Definition 3.** The *closeness operator* $\mathrm{cl}_{\mathcal{S}} : \wp X \to \wp X$ of a similarity space $(X, \mathcal{S})$ is defined by $\mathrm{cl}_{\mathcal{S}}(A) = \{x \in X : \forall \sim \in \mathcal{S}, \exists a \in A : a \sim x\}$.

The value of the closeness operator acting on a set $A \subseteq X$ is the collection of all elements arbitrarily similar to some element in $A$, i.e., those similar to an element of $A$ in all respects determined to be relevant by $\mathcal{S}$. The closeness operator of a similarity space is a generalization of the closure operator for a topological space: when restricted to sets of elements in the joint domain $\mathrm{cl}_{\mathcal{S}}(X)$ of the similarity structure, it satisfies the Kuratowski axioms for closure operators (Willard 1970, §3) except for idempotence, i.e., that for any $A \subseteq X$, $\mathrm{cl}_{\mathcal{S}}(\mathrm{cl}_{\mathcal{S}}(A)) = \mathrm{cl}_{\mathcal{S}}(A)$.

To understand the significance of these observations, it will be helpful to introduce some further definitions, both for topological and similarity structures. One way to characterize a topology on a set $X$ is to provide a neighborhood base for each of its elements (Willard 1970, Theorem 4.5).

**Proposition 1.** *Suppose that for each $x \in X$ there is assigned a set $\mathcal{B}_x \subseteq \wp X$ satisfying the following properties:*

1. *if $V \in \mathcal{B}_x$ then $x \in V$;*
2. *if $V_1, V_2 \in \mathcal{B}_x$ then there exists $V_3 \in \mathcal{B}_x$ such that $V_3 \subseteq V_1 \cap V_2$; and*
3. *if $V \in \mathcal{B}_x$, then there is some $V_0 \in \mathcal{B}_x$ such that for all $y \in V_0$, there exists some $W \subseteq V$ such that $W \in \mathcal{B}_y$.*

*Each such $\mathcal{B}_x$ is called a* neighborhood base *for $x$, and its elements* basic neighborhoods *of $x$. Then, if any $O \subseteq X$ is said to be* open *just when it contains a basic neighborhood of each of its points, the resulting set of open sets forms a topology for $X$.*

A neighborhood base for a point is thus a set of subsets containing that point that are (up to taking supersets) closed under intersection, and that are (allowing one to throw away some points therefrom) also basic neighborhoods of the other points they contain. Basic neighborhoods are also directly related to Kuratowski closure operators according to the following proposition (Willard 1970, Theorem 4.7c).

**Proposition 2.** *Given a subset $A$ of the set $X$ equipped with a neighborhood base $\mathcal{B}_x$ for each of its points $x$ and the topology resulting therefrom, the set $\{x \in X : \forall B \in \mathcal{B}_x, B \cap A \neq \emptyset\}$ is the closure of $A$ in that topology.*

To see the parallels with similarity structures, consider one further definition.

**Definition 4.** The *domain at* $x \in X$ of a similarity relation $\sim$ on $X$ is the set $D_x = \{y \in X : y \sim x\}$.

The domain of $\sim$ at $x \in X$ is just the set of all the elements in $X$ that are similar to $x$. Elementary logical manipulation of definitions 3 and 4 yields the following.

**Proposition 3.** *Let* $\mathrm{cl}_{\mathcal{S}}$ *be the closeness operator for a similarity space* $(X, \mathcal{S})$ *and* $\mathcal{B}_x$ *denote the set of domains at* $x \in X$ *of the similarity relations in* $\mathcal{S}$*. Then, for any* $A \subseteq X$, $\mathrm{cl}_{\mathcal{S}}(A) = \{x \in X : \forall D_x \in \mathcal{B}_x, D_x \cap A \neq \emptyset\}$.

The set of domains at $x \in X$ plays the same role in a similarity structure as a neighborhood base at $x$ does for a topological structure. Indeed, every basic neighborhood $B$ of a point $x$ determines a certain *minimal similarity relation* $\sim_B$ at that point: $y \sim_B x$ iff $B \in \mathcal{B}_x$ and $y \in B$. Conversely, any similarity structure on $X$ whose sets of non-empty domains at each $x$ satisfy conditions 2 and 3 of proposition 1—condition 1 is always satisfied by definition—determines a topology for $X$. Hence:

**Proposition 4.** *The set of minimal similarity relations determined from the basic neighborhoods of a topology is a similarity structure. Further, the domains of these minimal similarity relations themselves form a neighborhood base for the same topology.*

When a similarity structure does not determine a topology, condition 2 or 3 must be violated. Because condition 2 can always be satisfied by closing a similarity structure under intersection, at issue is really condition 3. This corresponds with the general failure of idempotence for the closeness operator of a similarity space, which allows one repeatedly to enlarge a set by considering elements similar to it—see also footnote 5.

For an example, consider the integers equipped with a single symmetric similarity relation that holds between any element and its successor (hence also its predecessor). The elements close to $\{0\}$ are $\{-1, 0, 1\}$, the elements close to those are $\{-2, -1, 0, 1, 2\}$, and so on. This isn't possible with a standard topology because topological basic neighborhoods must satisfy condition 3, leading to the discrete topology on the integers. Another example would be the real line equipped with similarity relations that hold between elements as long as their absolute difference is below a certain value $\epsilon$. This might represent a kind of indistinguishability among sufficiently similar realizations of a quantitative property. But this also isn't possible to represent with a standard topology for the same reasons as the previous example.

There are no restrictions on the kinds of entities represented by the elements of $X$. They could be objects, states of affairs, or even values of a property. So, a similarity structure can represent how a variety of objects, states of affairs, or values of a property are similar, including but not limited to when that structure is topological. The aforementioned elements, or *models* of these entities, are typically themselves highly structured, with sub-objects, relations among them, and various other properties and pieces of architecture. For instance, a spacetime model has properties—e.g., what temporal structure it supports (cf. section 4.1)—but also sub-objects such as smooth curves. These in turn have properties—e.g., being timelike or not—and sub-objects, viz., spacetime points. In what follows in sections 3 and 4, I will be interested in the properties of the models themselves, but also the properties of their sub-objects, the values all these properties can take on, and so on. A bit more formally:

**Definition 5.** A *property assignment* (or *valuation*) on a collection $X$ is a (perhaps partial) map $\nu : X \nrightarrow V$, where $V$ is called its space of *(property) values.* Further, if $x \in X$ is (not) in the domain of $\nu$, then the property represented by $\nu$ is said to be *(not) defined for* $x$.

A property assignment represents a property intensionally, as map from models to values. For properties of sub-objects, instead of tediously and explicitly composing a property assignment with a map picking out a sub-object of a model (or some more complex such composition), I shall often talk about and represent them simply as property assignments to the models themselves. One need just take care to ensure that the particular sub-object is well-defined.

Philosophers will already be familiar with (classical) propositional properties, those which either obtain or not for a given object or state of affairs. For example, consider a collection of paint color samples, whose various pigment properties we represent in a corresponding collection of models. There is a propositional property, "is red," that applies to them, and has the Boolean domain $\{\top, \bot\}$ as its values.

Property assignments can also take on quantities as their values. For example, consider free Hamiltonian systems of a single particle. Although Hamiltonian systems are complex, having many sub-objects and structures (e.g., symplectic structure), we can pick out the parts (e.g., the appropriate term in the Hamiltonian) from each one that describe the mass of the particle each represents. Thus there is a property assignment representing "has mass $m$" to such models, whose space of values is the positive real interval: $m \in (0, \infty)$. There is also a property assignment representing "has velocity $v$ (with respect to a certain reference frame $F$) at time $t$," whose space of values is a three-dimensional real vector space, if the reference frame and time are taken as metavariables to be filled in with some constant, and a more complex structure if not. It is important to accommodate quantitative and complex properties, as they are essential for evaluating the predictive success of many scientific models.

Property assignments to models can still pick out properties of sub-objects and structures when there are many of them. Consider again a Hamiltonian system of countably many classical, distinguishable massive particles, and suppose (without loss of generality) that the particles have been labeled by a natural number. Then there are property assignments representing "particle 1 has mass $m_1$," "particle 1 has velocity $v_1$ (with respect to a certain reference frame $F_1$) at time $t_1$," "particle 2 has mass $m_2$," "particle 2 has velocity $v_2$ (with respect to a certain reference frame $F_2$) at time $t_2$," and so on.

There are no restrictions on what kinds of systems and properties thereof the models and assignment maps respectively represent, including those represented by sub-objects and properties thereof in the models. Any property assignment definable for the elements of $X$ (and their sub-objects), broadly understood, is a candidate for consideration.[4] This includes properties most naturally defined by functional composition. For instance, the disjunction of a collection of properties with Boolean values yields another property with Boolean values. And the kinetic energy of a simple particle in Newtonian physics, $K = \frac{1}{2}m|v|^2$, is an algebraic function of its mass $m$ and velocity $v$. Properties whose assignments are merely partial must have conditional definitions, of course, and expansions of such properties are also candidates for

[4] I am being liberal and informal about definability here since the details do not matter significantly for my purposes; for more on definability, see, e.g., Chang and Keisler (1973), Tuomela (1973), Barwise (1975), and Rantala (1977).

consideration. Whether a given such property is *interesting* is another matter. I discuss its relevance for emergence further in section 3.

# 3 Emergent Properties

Using the framework of similarity structures introduced in section 2, one can define the four types of emergence for properties mentioned in section 1. Each of them explicates the novelty of a property of a model, relative to some comparison class, in a slightly different way. I first state the formal definitions of these concepts, then discuss their interpretation, informal conditions of application, relations, and what makes particular examples of them interesting.

Before doing so, it will be helpful to note a point of terminology. The convention of common parlance is to talk of the potential emergence of properties, without reference to an assignment function representing which models have which properties. It will facilitate the technical aspects of my presentation to refer instead at times to emergent property *values*, as I have already occasionally done, rather than the properties simpliciter. But this should be understood as just a formal way of representing the common parlance: an assignment of a value to a model is merely a representation of the model's properties (which is in turn perhaps a representation of some phenomenal or ontological properties). So, to say that some collection of property values is emergent is just to say that the particular properties those value assignments represent are emergent.

Suppose, then, that a collection of scientific models $X$ has been equipped with a similarity structure $\mathcal{S}$, as has the space of property values $V$ with a similarity structure $\mathcal{V}$ for some property assignment $\nu : X \nrightarrow V$.

**Definition 6.** With respect to the models $A \subseteq X$, the values $\nu[B]$ of a collection of models $B \subseteq X$ under the assignment $\nu$ are said to be:

1. *weakly emergent* when $\nu[B] \nsubseteq \nu[A]$;
2. *strongly emergent* when $\nu[B] \nsubseteq \mathrm{cl}_{\mathcal{V}}(\nu[A])$;
3. *non-reductively emergent* when $\nu[B] \nsubseteq \nu[\mathrm{cl}_{\mathcal{S}}(A)]$; and
4. *radically emergent* when $\nu[B] \nsubseteq \mathrm{cl}_{\mathcal{V}}(\nu[\mathrm{cl}_{\mathcal{S}}(A)])$.[5]

For each of these, it is said to obtain *completely* when both sides of its non-inclusion are non-intersecting. Otherwise, it may said to obtain *partially*.

Note that while standard discussions of emergent properties restrict attention to a property of a single model, the above definitions as stated apply to the property values for sets of models. But, when they are so restricted, partial and complete emergence become equivalent.

The weakest of the above, weak emergence, requires only the mere non-identity of the values of the property in question, $\nu[B]$, with any of those for the models in the comparison class, $\nu[A]$. As Teller (1992, 140) complains, however, an object's property of

[5]Beyond these latter three, one can define infinitely many concepts of emergence based on iterations of the closeness operators: with respect to the models $A \subseteq X$, the values $\nu[B]$ of a collection of models $B \subseteq X$ can be said to be $(n, m)$-*emergent* when $\nu[B] \nsubseteq \mathrm{cl}^n_{\mathcal{V}}(\nu[\mathrm{cl}^m_{\mathcal{S}}(A)])$. However, if the similarity structures on the spaces of models and property values have idempotent closeness operators, then the above four concepts are at most the ones that are distinct.

having a mass of 10.74218 grams would be merely novel in this way with respect to the comparison class of objects without that mass, which seems uninteresting. Thus one might demand not just mere non-identity, but a sort of unexpectedness or comparative unexplicability. This can be formalized along at least two dimensions, captured by strong and non-reductive emergence. The strong emergence of a property of a model requires that the property must also be not sufficiently similar to the properties of the models of the comparison class—it is unexpected because it is not even similar (in the relevant ways) to the properties available for consideration from the comparison class. This requires similarity structure on the space of values that a property can take on. The non-reductive emergence of a property of a model requires that the property must also be non-identical with the corresponding properties of the models arbitrarily similar to those in the comparison class. This requires similarity structure on the joint collection of models. Finally, one can combine these criteria in radical emergence: the emergent property is not even sufficiently similar to the properties of the models arbitrarily similar to those in the comparison class. Accordingly, this requires similarity structure on both the space of values that a property can take on and the joint collection of models themselves.

These interpretations are of a set of formal definitions, but there is an important informal condition of application for them. Generally, models $A$ whose putative emergent properties one is considering, as well as its comparison class, $B$, should have *overlapping representational domains*: that a property of avalanches, for example, is inexplicable from models of radioactive decay is hardly surprising or interesting, and not even deserving of the name emergence.[6] Moreover, the only types of properties that are

[6] It won't do to formalize this requirement simply as the nonempty intersection of $A$

candidates for being emergent are those that the models aim to represent. This is not a formal requirement, as the structure of a model itself does not determine what the model represents. For instance, instrumentalist interpretations of the models of classical and quantum mechanics will permit only emergent empirical properties, while realist interpretation could permit more.

This suggests another condition of application: I require that the comparison class $B$ should be *at least as fundamental* as the class $A$, at least with respect to properties they aim to represent and which are considered as candidates for being emergent. This requirement distinguishes even weak emergence from mere non-identity and captures the idea that emergent properties must "arise" from a more fundamental base. I do not here take a stand on any particular account of relative fundamentality, in part because the sort needed will depend on the aims of the models' representations. Some realist-oriented possibilities include supervenience or composition between different scales or levels of description, reality, or explanation (List 2018).[7] An anti-realist-oriented possibility could be empirical adequacy (van Fraassen 1980): one set of models is at least as fundamental as another when it is at least as empirically adequate for at least the phenomena under consideration. Although I have left this condition of application informal, these possible explications could likely be developed in formal detail, but this is not necessary for

and $B$ if these are understood as sets of models in the sense found in model theory: this intersection will not in general be nonempty, as all the examples of section 4 demonstrate. Conversely, these sets might be entirely overlapping, yet have no overlapping representational domains, as the many applications of the simple harmonic oscillator attest (Fletcher 2018b).

[7]For further metaphysical accounts, see Tahko (2018) and references therein.

present purposes.

For models used in science, this last notion of relative fundamentality and the particularity of the properties represented seems to be especially important. To the extent that the candidate emergent property values are realized within a strict subset of the models of a theory, only those models will be empirically adequate representations of those property values. Even if some sort of empirical adequacy is an informal condition of application, one can still provide arguments about hypothetical emergence when such adequacy is unknown, or counterfactual emergence when such adequacy is known to be false. This is important for capturing emergence claims in frontier science, such as quantum gravity, where a correct empirical description of the phenomena has yet to be established (Crowther 2016).

In any case, each of these types of emergence can be understood as epistemological, in the sense that, from the perspective of the comparison class, the emergent property in virtue of its novelty cannot be deduced. Whether any type is ontological or metaphysical will again depend on whether one takes a realist attitude towards the models, and the properties they represent.

The different types of emergence bear various logical relations to one another. Radical emergence entails the other three types of emergence, while weak emergence is entailed by the other types. In general, neither strong emergence nor non-reductive emergence entails the other. And in each of these cases, emergence can obtain either because the property assignments are defined for the models of $A$ but are different for the models of $B$, or they are not defined for the models of $A$. In the latter case, I suggest one could deem it a case of conceptual emergence, since the emergent property itself is novel, not just its values. Conceptually emergent properties are more plausibly *interesting* cases of emergence (cf. Teller 1992, 140–41) since, for strongly emergent properties, at least, they are so independently of the similarity structure on the property's space of values.

I cannot provide a list of necessary and jointly sufficient criteria for an emergent property to be interesting—indeed, I am skeptical that such a list could be feasibly produced. But all else being equal, the emergent property may be more interesting to the extent to which is it more robust, which "means something like: 'the same for various choices of, or assumptions about, the comparison class' " (Butterfield 2011a, 921). In other words, an emergent property of a model is interesting to the extent to which it is so for a relatively wide comparison class of other models. Many artificially defined properties, such as some "mere Cambridge properties" (Francescotti 1999), will be uninteresting on this criterion because they do not hold when one varies assumptions about the comparison class that one has antecedent reason to believe are irrelevant.

Similar reasoning extends to conditionally defined properties whose domain of definition—the models on which the property assignment is defined—is needlessly restricted. An example would be the kinetic energy of a simple particle in Newtonian physics, but restricted to particles of mass within a certain finite range. Expansions of conditionally defined properties—that is, of their assignment's domain of definition—will also be uninteresting to the extent that they are deemed unprincipled. For instance, the stress and strain of a body modeled as a material continuum are the internal distribution of forces within the body and the deformation of the body from a reference (usually equilibrium) configuration. It would be unprincipled then to extend this definition to other materials (e.g., liquids and gases) by assigning them stresses and strains equal, e.g., to their energy or temperature.[8] Here the requirement of overlapping

[8]Determining how to extend a concept in a principled way is not typically trivial, and

representational domains provides additional constraint because the terms defined can have a meaning given in part externally, e.g., through the means of empirical testing. Through these means one can often identify chimerical properties and judge whether their emergence is interesting.

One further positive criterion that makes an example of emergence interesting is whether it facilitates novel explanations (Knox 2016). Each of the types of emergence defined involved a sort of comparative novelty in the values of a property found in a set of models with respect to a comparison class. Thus, models with emergent properties provide new resources for model-based explanations that cannot be found merely by abstracting away details from the models of the comparison class.

All such conditions on what make an example of emergence interesting are in a sense necessary concessions to what Humphreys (2016) calls the *rarity heuristic* for emergence: a satisfactory account of emergence should not make cases thereof too common. I have adapted the heuristic, though, not to emergence itself but to its interesting examples. So, although emergence may be fairly common on the above account, interesting examples of it are less so. While some may take this is as a conventional matter of shifting content from one definition to another, I find it useful to separate the relatively precise formal structure that allows us to make attributions to properties described by models and theories from how those attributions figure in broader intellectual and scientific projects. In other words, a property's emergence, by itself as a partly formal property, does not portend significance, and we should recognize that such significance

I cannot provide here a general theory of how to do it or when it is possible. See Wilson (2006) for many examples, including extended analysis of the "hardness" concept and its space of values in many contexts.

may be grounded in concerns external to those which determine how, say, various models are relevantly similar to one another. The context of investigation determines the similarity structure on a space of models, which in turn determines which properties are emergent with respect to others, but which of those emergent properties are interesting may depend on much more than this.[9]

Before illustrating this account of emergence in the next section, let me acknowledge that, even constrained by the informal conditions of application, it may strike some readers as *too* weak to count as being of emergence rather than of mere dissimilarity—especially weak emergence. As I have stated myself, weak emergence requires very little, so some readers may wish to take it only as a precondition for more interesting forms of emergence. I invite them to consider even further conditions of application under which it would deserve the name, but I shall proceed in what follows without repeated qualifications.

[9]I'm tempted to say that emergence is the objective core of the partly subjective concept of interesting emergence, but think this would misleadingly suggest that interesting emergence arises only from the addition of contextual and relational features, when in fact emergence itself already includes these—see section 5. Rather, the added features are contextual of and relational to ideas and interests that go beyond those that determine relevant similarity, such as those pertaining to causal reasoning and explanation, or suggest novel and fruitful avenues for future research.

# 4 Examples

In this section I illustrate two applications of the similarity structure approach to emergence. The first, in section 4.1, shows how properties related to absolute simultaneity are emergent in classical mechanics relative to special relativity—that is, in models of Galilean spacetime relative to models of Minkowski spacetime. The second, in section 4.2, concerns the emergence of phase transitions in certain thermostatistical systems, represented as nonanalytic points in the system's free energy, under the thermodynamic limit—that is, when one considers infinite versions of the systems in question.

## 4.1 Simultaneity

One of the hallmarks of relativity theory is that it makes the simultaneity of events relative to an observer: different observers may determine different classes of events to be simultaneous with one another.[10] By contrast, classical spacetime physics includes a concept of objective simultaneity because all observers agree on these classes. Moreover, special relativity is the more fundamental of the two about spacetime physics. So, in comparing the models of the two theories, we may assess whether objective simultaneity is emergent, and in what sense. Let $T_M$ ("M" for "Minkowski") be the models of special relativity and $T_G$ ("G" for "Galileo") be the models of classical mechanics, which are disjoint. Further, let $\nu_s : T_M \cup T_G \to \{\top, \bot\}$ be the property assignment whose value is

[10] In Minkowski spacetime—the domain of special relativity—observers always agree that timelike-related events are not simultaneous with one another, but may determine different sets of spacelike-related events to be simultaneous.

$\top$ when evaluated on a model with a concept of objective simultaneity, and is $\bot$ otherwise. (We could also consider more specific properties, such as particular definite temporal relations between events, which may not be defined for the models in $T_M$, but this will not matter for the broad point at hand.) Clearly, then, objective simultaneity is weakly emergent in the models of classical mechanics with respect to the models of special relativity because $\nu_s[T_G] = \{\top\}$ and $\nu_s[T_M] = \{\bot\}$ are disjoint.

Is objective simultaneity also strongly, non-reductively, or radically emergent? Answering these questions requires justifying a choice of similarity structure on the property value space $\{\top, \bot\}$, the space of models $T_M \cup T_G$, or both, respectively. To begin with the question of strong emergence, a quite natural similarity structure $\mathcal{V}$ for the property value space would consist just in the relation $\sim_T = \{(\top, \top), (\bot, \bot)\}$, which asserts that only (not) having a propositional property is similar to (not) having that property, i.e., "true" is only similar to "true," and "false" only to "false." It follows that $\mathrm{cl}_{\mathcal{V}}(\{\bot\}) = \{\bot\}$, hence objective simultaneity is strongly emergent in the models of classical mechanics with respect to the models of special relativity.

As for the question of non-reductive (and radical) emergence, we can use the work of Fletcher (2016, 2019b), who describes two kinds of topologies on the set of spacetimes, both relativistic and nonrelativistic, that can be interpreted as similarity structures on the basis of proposition 4. One of these, the compact-open topology, determines spacetimes to be similar just when their empirical predictions for observables—real scalar quantities constructed from the spacetime metric(s), their derivatives, and frame fields—on a compact (i.e., bounded) region of spacetime are numerically close. Here, the similarity relations vary over all the compact regions and all degrees of closeness within the positive real interval. The other of these, the open topology, determines spacetimes to be similar in much the same way, but over the whole of spacetime, not just compact regions. (See appendix A.1 for a more complete description.) Fletcher (2019b) shows that according to the former, $T_G \subseteq \mathrm{cl}_{\mathcal{CO}}(T_M)$, but according to the latter, $T_G \cap \mathrm{cl}_{\mathcal{O}}(T_M) = \emptyset$. Hence $\nu_s[T_G] \subseteq \nu_s[\mathrm{cl}_{\mathcal{CO}}(T_M)] = \{\top, \bot\} = \mathrm{cl}_{\mathcal{V}}(\nu_s[\mathrm{cl}_{\mathcal{CO}}(T_M)])$, meaning objective simultaneity is neither non-reductively nor radically emergent according to the former. But since $\nu_s[\mathrm{cl}_{\mathcal{O}}(T_R)] = \{\bot\}$, objective simultaneity is radically emergent according to the latter.

Which of these is the "right" verdict? Is objective simultaneity radically emergent, or merely strongly emergent in classical spacetimes? There is in general no context-free answer. If the goal is a description of the properties of classical spacetimes that are novel compared with those of relativistic spacetimes in light of the former's success in actual empirical descriptions, then there is good reason to believe that the compact-open topology is a better choice than the open topology, since our actual data are confined to bounded spacetime regions. But if the goal is such a description in light of a "global view" of the features of spacetime, regardless of whether they are reflected in what is observable, then the open topology may be better.

## 4.2 Phase Transitions

Phase transitions are sudden qualitative changes in the characteristics of a highly composite system, such as a drop of water freezing or an iron bar magnetizing. While there is a plethora of approaches to understanding phase transitions and philosophical debates surrounding it (Menon and Callender 2016), for simplicity of exposition I will focus on a crude toy model, in which a candidate (equilibrium) thermostatistical system is represented through, e.g., a many-body Hamiltonian. The energy levels for the components determine energy levels for the total system, which in turn determine the (canonical) partition function for the system and its (Helmholtz) free energy function. A phase transition for the system occurs when the system enters into state whose free energy is a nonanalytic point—i.e., a point at which some order of its partial derivatives do not exist. Although having such a point (or not) is a property of a function, which is itself a derived property of a model of the system, for the reasons discussed in section 2 I will refer to it as a property of the model itself.

Let us focus, without loss of generality, on a certain type of thermostatistical model (say, the two-dimensional Ising model, but the details will not concern us). Let $T_F$ be the models with finitely many components and $T_\infty$ be those with infinitely many, each element of which is represented by the (toy) free energy function of a single variable $F_n(x) = 1/(1 + e^{-nx})$, where $n$ is the number of components and $x$ is the thermodynamic variable.[11] Meanwhile

$$F_\infty(x) = \begin{cases} 0, & x < 0, \\ 1/2, & x = 0, \\ 1, & x > 0, \end{cases}$$

the Heaviside step function. For real systems, the models with finitely many components are more fundamental because real systems are finite in just this way; the ones with infinitely many components are idealizations.

Further, let $\nu_a : T_F \cup T_\infty \to \{\top, \bot\}$ be the property assignment whose value is $\top$

[11] Cf. the toy examples provided by Butterfield (2011b, 1078–79) and Norton (2012, 224).

when evaluated on a model that is analytic, and is $\bot$ otherwise—the former evinces no phase transitions, while the latter does. Clearly $\nu_a[T_F] = \top$ while $\nu_a[T_\infty] = \bot$. (We could also consider more specific properties, such as particular definite phase transition properties, which would not be defined for the models in $T_F$, but this will not matter for the broad point at hand.) Thus phase transitions are weakly emergent for these crude toy models with infinitely many components with respect to the ones with finitely many. In fact, for the same reasoning as the simultaneity case described in section 4.1, they are also strongly emergent.

To answer whether they are also non-reductively or radically emergent, one must fix a similarity structure on the models $T_F \cup T_\infty$. Two function space topologies that can indeed be interpreted as similarity structures are the topologies of pointwise and uniform convergence, respectively (Willard 1970, §42). The topology of pointwise convergence determines functions to be similar just when they are numerically close on a finite number of points. Here, the similarity relations vary over all such finite collections and all degrees of closeness within the positive real interval. The topology of uniform convergence, meanwhile, determines functions to be similar in much the same way, but over the whole of real line, not just at finite points. (See appendix A.2 for a more complete description.) It's elementary to show that $T_\infty \subseteq \mathrm{cl}_{\mathcal{PC}}(T_F)$ while $T_\infty \cap \mathrm{cl}_{\mathcal{UC}}(T_F) = \emptyset$. So, by the same reasoning as in section 4.1, phase transitions are neither non-reductively nor radically emergent by pointwise convergence, but are radically emergent by uniform convergence.

Which of these is the "right" verdict? Are phase transitions radically emergent, or merely strongly emergent in infinite systems? Again, there is in general no context-free answer. If the goal is a description of the properties of infinite systems that are novel compared with those of finite systems only in light of the former's success in actual empirical descriptions, then there is good reason to believe that the topology of pointwise convergence is a better choice than that of uniform convergence, since our actual data of the free energy functions of systems are confined to point measurements (Callender 2001). But if the goal is such a description in light of a "global view" of the features of thermostatistical systems, regardless of whether they are reflected in what is observable, then the topology of uniform convergence may be better.

## 5 Features

Both of the examples of emergence in section 4 arise from a kind of infinite idealization: with respect to a more fundamental comparison class, the models with the emergent properties take some other property value to be infinite or infinitesimal, which allows for emergence. Although these are common in physics—see Fletcher et al. (2019) for more examples—they also appear in other scientific disciplines. For example: in biology one might describe the emergence of *deterministic* genetic transmission in certain population genetics models compared with stochastic ones by introducing an infinite population (Abrams 2006; Strevens 2019); in economics one might describe the emergence of unbounded rationality in models of agents with infinite patience, perfect homogeneity of choices, or undergoing infinite game-theoretic interactions (Brennan et al. 2008); and in linguistics, one might describe the emergence of languages with infinite expressions by abstracting away the material constraints on mental language representation (Nefdt 2019).

In any case, one of the features of the current account of emergence concerns its relation with reduction. Both of the examples of emergence in section 4 could be compatible with reduction when the latter is understood as the models of one theory being arbitrarily similar to the models of another, i.e., $T$ reduces $T'$ when $T' \subseteq \mathrm{cl}(T)$.[12] In this case, both the weak and strong emergence of a property in a model with respect to a comparison class are compatible with that model being arbitrarily similar to the members of the comparison class, but both non-reductive and radical emergence are not. So, some types of emergence represent a sort of relative dissimilarity that vindicates non-reductive intuitions concerning emergence, while some do not.

Which types of emergence (outside of the weak variety) pertain to a property of model, if at all, depends as well on how the model and its comparison class are relevantly similar to one another. I am skeptical that there will be a fixed similarity structure that accrues to a theory or collection of models once and for all, for similar high-level reasons as those Fletcher (2016) adduces in the case of general relativity. In that case, an explanatory question determines an investigative context. The properties of models that make a difference for answering the explanatory question are precisely those that are relevant to this context. Fletcher (2016) shows that different such questions—regarding the stability of certain temporal properties, on the one hand, and the deterministic nature of temporal evolution, on the other[13]—determine different enough such contexts that amalgamating them would yield absurd answers for at least one.

[12] Of course, much more can be said about this relation, its relation to other notions of intertheoretic reduction, its applications, etc., but space does not permit that here—see Fletcher (2018c) for a fuller account.

[13] More technically, the questions regard which relativistic spacetimes are stably causal, and which have a well-posed initial value problem.

My suggestion for how to guide the selection of similarity structure in adjudicating cases of emergence is similarly contextual. In such one is often interested in the explanatory resources one sort of model has with respect to another—the former with the putative emergent property with respect to the latter. Again, the properties of models that make a difference for answering the explanatory question are precisely those that are relevant to this context, so models are similar to the extent that they have similar such relevant properties. So, similarity in the space of models can be guided by similarity in the space of their relevant property values.[14] Often these values will represent empirical predictions, in which case the measurement theory for the property can often provide guidance as to the relevant similarity structure on the space of values. Boolean and real-valued property assignment, for example, would tend to have the same usual similarity structure as discussed earlier in this essay. Because the range and precision of these predictions can vary over time and between researchers according to what scientists can and have measured, and how and what they take the models in questions to represent, whether a property is emergent depends as well on these factors, which are not a part of the theory or its formalism. So, even though the definitions of emergence I provide are partly formal, they require input from outside the formalism itself.

That input, of course, can come in more or less precise forms. The extent to which the context of investigation can be made precise by determining the properties of models that make a difference to the questions being asked (and no more) is the extent to which the similarity structure can be made precise. And the extent to which the theory or models to which one is restricting attention can be made precise is the extent to which

[14]Cf. the more detailed approach to determining similarity among models in Fletcher (2019a), albeit for the different purpose of counterfactual reasoning with them.

the resulting similarity space can be made precise. For physical theories, whose models and properties are already mathematized, this may be just a matter of technical ingenuity, but for other sorts of vaguer theories or models, this may require deliberate and determined investigation into how they can be made precise. I take this not as a vice of the similarity structure approach to emergence but a virtue: without offering false precision, it forces one to reveal, inasmuch as one can, one's commitments to details hitherto obscured about just what questions one is asking of just which models or theories.

# A Similarity Structure from Topologies

## A.1 The Compact-Open and Open Topologies

Section 4.1 invokes the compact-open and open topologies on relativistic and nonrelativistic spacetime structures, and asserts that they generate similarity structures. In this subsection, I describe in more detail these topological and similarity structures.

Formally, for any $(r, s)$-tensor field $K^{a_1 \cdots a_r}_{b_1 \cdots b_s}$ on a smooth manifold $M$ endowed with a smooth Riemannian metric $h$, one can define its $h$-fiber norm (Fletcher 2018a) to be the scalar field

$$|K^{a_1 \cdots a_r}_{b_1 \cdots b_s}|_h = |K^{a_1 \cdots a_r}_{b_1 \cdots b_s} K^{c_1 \cdots c_r}_{d_1 \cdots d_s} h_{a_1 c_1} \cdots h_{a_r c_r} h^{b_1 d_1} \cdots h^{b_s d_s}|^{1/2}.$$

This is the Frobenius norm of $K^{a_1 \cdots a_r}_{b_1 \cdots b_s}$ as computed in a basis in which $h$ is the identity matrix. (It turns out that the particular norm one uses makes no essential difference for the resulting topological (and similarity) structure induced (Fletcher 2018a), but expressing the structure in these terms is relatively simple.) For any two tensor fields of the same rank, such as two (temporal or spatial) metrics or their derivatives (with respect to the Levi-Civita connection of $h$), one can describe the magnitude of their dissimilarity across $M$ as the $h$-fiber norm of their difference. For example, any two $(0, 2)$ tensor fields $g, g'$ representing (temporal) metrics of spacetimes could be regarded as $(\epsilon, h, C)$-similar just when the supremum of that magnitude over a compact $C \subseteq M$ is no more than $\epsilon$. This determines a local neighborhood base $\mathcal{B}_g$ for the compact-open topology for such fields and similarity relations for them as well:

$$\mathcal{B}_g(\epsilon; h, C) = \{g' : \sup_C |g - g'|_h < \epsilon\},$$

$$\sim_{(\epsilon,h,C)} = \{(g', g) : \sup_C |g' - g|_h < \epsilon\}.$$

The basic neighborhood system and the similarity structure are formed by letting $g$ range over all tensor fields of the same rank (perhaps restricted further), $\epsilon$ over all positive real numbers, $h$ over all Riemannian metrics, and $C$ over all compact sets.

The basic neighborhood system and the similarity structure for the open topology are formed in the same way, except that $C$ ranges over all *open* sets in the manifold topology, including $M$ itself.

## A.2 The Topologies of Pointwise and Uniform Convergence

Section 4.1 invokes the topologies of pointwise and uniform convergence on real functions of a single variable, and asserts that they generate similarity structures. In this subsection, I describe in more detail these topological and similarity structures.

Formally, we may compare (total) real functions $f, f'$ by the absolute difference in their values on common points of their domain. (It turns out that the particular norm one uses makes no essential difference for the resulting topological (and similarity) structure induced.) A local neighborhood base for the topology of pointwise convergence, and its corresponding similarity structure, are given by

$$\mathcal{B}_f(\epsilon; x) = \{f' : |f'(x) - f(x)| < \epsilon\},$$
$$\sim_{(\epsilon,x)} = \{(f', f) : |f'(x) - f(x)| < \epsilon\},$$

where $f$ ranges over all such functions, $\epsilon$ over all positive real numbers, and $x$ over all real numbers. It is called the topology of pointwise convergence because, according to it, a sequence of functions $f_\lambda$ converges to a function $f$ if and only if the sequence of numbers $f_\lambda(x)$ converges to the number $f(x)$ for all values of $x$ (Willard 1970, Theorem 42.2).

The topology of uniform convergence is determined similarity, except that real functions are compared not by their values at individual common points, but by the supremum thereof. Formally, then, a local neighborhood base for the topology of uniform convergence, and its corresponding similarity structure, are given by

$$\mathcal{B}_f(\epsilon) = \{f' : \sup_{x \in \mathbb{R}} |f'(x) - f(x)| < \epsilon\},$$
$$\sim_{(\epsilon)} = \{(f', f) : \sup_{x \in \mathbb{R}} |f'(x) - f(x)| < \epsilon\},$$

where $f$ ranges over all such functions and $\epsilon$ over all positive real numbers. It is called the topology of uniform convergence because, according to it, a sequence of functions $f_\lambda$ converges to a function $f$ if and only if the sequence of numbers $\sup_{x \in \mathbb{R}} f_\lambda(x)$ converges to the number $\sup_{x \in \mathbb{R}} f(x)$.

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