Quantum Indeterminacy and the Eigenstate-Eigenvalue Link

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Abstract: Can quantum theory provide examples of metaphysical indeterminacy, indeterminacy that obtains *in the world itself*, independently of how one represents the world in language or thought? We provide a positive answer assuming just one constraint of orthodox quantum theory: the eigenstate-eigenvalue link. Our account adds a modal condition to preclude spurious indeterminacy in the presence of superselection sectors. No other extant account of metaphysical indeterminacy in quantum theory meets these demands.

1 Introduction: Metaphysical Indeterminacy and Quantum Theory

Metaphysical indeterminacy—that is, indeterminacy that obtains *in the world itself*, independently of how one represents the world in language or thought—has recently captured the attention of metaphysicians. Initial examples centering on vague predication and objects have been inspiring but controversial (Akiba 2004; Rosen and Smith 2004; Williams 2008b; Barnes 2010). This has brought new focus to a relatively neglected dimension of the metaphysics of quantum theory: it may provide a class of exemplars of metaphysical indeterminacy less burdened with the complexities of natural language and intuitions about everyday objects, and in that way furnish a test of any general account of metaphysical indeterminacy (Williams 2008b; Wilson 2013; Bokulich 2014).[1] Our goal in this essay is to develop a model of metaphysical indeterminacy that can be successfully applied to the quantum case.[2]

Any complete account of quantum indeterminacy must coordinate three different parts: the physical models of quantum systems, the logic of indeterminacy for those models, and the metaphysical interpretation of that logic. The account we offer here takes as its starting point the physical models. We make only one substantive assumption about such models, namely that they

[1] That said, not all commentators believe that quantum theory must provide such examples (Glick 2017).

[2] A clarification is in order regarding our gloss on "metaphysical" indeterminacy as applied to quantum theory. When we say that such indeterminacy obtains independently of our linguistic and mental representations, we mean that it is not the result of, for example, semantic indecision or any other imprecision in our language or thought (Williams 2008b). We do *not* mean that it is indeterminacy that obtains independently of our interpretation or mathematical formalization of quantum mechanics, e.g., Bohmian mechanics, spontaneous collapse theories, the Many Worlds interpretation, etc. (See Myrvold (2018, sec. 4.2) and Lewis (2016, chap. 4) for a discussion of these interpretations and others.) Each of the latter takes a stance on how to correctly understand quantum theory, in particular the eigenstate-eigenvalue link—see §2.1 below—and thus represents a substantive position about what the world is ultimately like. On some of these interpretations there may be indeterminacy in the world itself—i.e., metaphysical indeterminacy—whereas on others there may not be.

are constrained by the eigenstate-eigenvalue link (EEL), which defines the exact conditions under which a quantum system has a property. Though we will provide some motivation for EEL later on, it is not our aim to argue for it here. Instead, our goal is to see what sort of theory of quantum indeterminacy arises by taking EEL as our starting point. And this seems justified: EEL is a central part of orthodox quantum theory, and it is common ground among most recent accounts of quantum indeterminacy (Calosi and Wilson 2019; Torza 2020; Darby and Pickup 2021).[3]

Though EEL is our only official assumption, it quickly leads us to embrace a second part of orthodox quantum theory, namely the interpretation of quantum logic as the logic of quantum property ascriptions. This way of interpreting quantum logic, though not strictly mandated by EEL, is a natural logic to accompany EEL.

Our account of quantum indeterminacy naturally emerges from assuming EEL and this interpretation of quantum logic. The indeterminacy we are concerned with here is located in the *instantiation of properties* by a quantum system (as opposed to, say, the quantum state of the system). Our account thereof has two components. First, every case in which it is indeterminate whether a quantum system has property *F* manifests itself in a *truth-value gap* with respect to the proposition ascribing *F* to that system. This part of our account follows immediately from the combination of EEL and quantum logic. Second, for any such case of quantum indeterminacy it must be *possible* for the aforementioned proposition to be true. We intend this modal proviso to rule out spurious cases of indeterminacy that would arise for property ascriptions making category mistakes, such as attributing a particular spin direction to a spin-0 system, which by definition has no net spin. All together, we take these two components to provide necessary and jointly sufficient conditions for the phenomenon of quantum indeterminacy.

In contrast with our own account, other accounts of metaphysical indeterminacy so far have been largely motivated, first, by concerns endogenous to the metaphysics literature and, second, by the desire to preserve classical logic, inasmuch as is possible. Nevertheless, their advocates often argue that these accounts can capture indeterminacy in orthodox quantum theory, and in particular while respecting EEL. In the second part of our essay, we argue to the contrary that none of the accounts by Calosi and Wilson (2019), Barnes and Williams (2011), Darby and Pickup (2021), and Torza (2020) is compatible with EEL.[4] These accounts might be successful on others terms—i.e., if they reject EEL and aim for an account of indeterminacy in some non-orthodox version of quantum theory—but in that case they would no longer be direct competitors to the

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[3] While it is not our intention in this essay to offer any conclusive argument in support of EEL, in §2.1 we will review the standard motivation for the principle and provide some evidence for our claim that EEL is in fact part of orthodox quantum theory. Not everyone agrees that EEL should be considered part of the orthodoxy: see Wallace (2012a, 4580; 2012b, 108; 2013, 215), but also Gilton (2016) for a rejoinder. Be this as it may, the fact that most recent accounts of quantum indeterminacy assume EEL is sufficient to warrant assuming it for present purposes.

[4] Barnes and Williams (2011) do not discuss quantum theory explicitly, although Williams does in an earlier review (2008b); the former has been influential for other accounts. Torza (2020, 4528–4529) endorses EEL in so many words, although not by name; Calosi and Wilson (2019, sec. 2.1) do so, too, for the purposes of most of their project; and while Darby and Pickup do not mention EEL, their analysis of possible quantum indeterminacy begins from the orthodoxy about quantum theory that presupposes EEL, i.e., a standpoint prior to sophisticated interpretation (2021, 1686, 1689).

present account. Along the way, in §4.2.4, we point out that once one accepts EEL, considerations stemming from the Kochen-Specker theorem don't add much to the case for metaphysical indeterminacy in quantum theory, contrary to a theme in the literature (Darby 2010; Skow 2010).

We end with a brief discussion of some questions left open by the account we defend here. One such question concerns the metaphysical interpretation of the logic of quantum indeterminacy. We suspect that any such interpretation will depend on more substantive interpretative assumptions than EEL for quantum theory. We similarly leave open the question of whether and, if so, how our account could be extended to encompass the phenomenon of metaphysical indeterminacy in other domains. Finally, we raise some open questions concerning extending our account of indeterminacy to more sophisticated forms of quantum theory involving mixed states and positive operator-valued measures.

2 The Eigenstate-Eigenvalue Link and Quantum Logic

2.1 Towards the Eigenstate-Eigenvalue Link

We assume the standard framework for elementary quantum theory of finite-dimensional Hilbert spaces as representing the possible states for quantum systems.[5] In particular, the set of rays in the Hilbert space represents the possible *states* of the system, and the set of self-adjoint operators $\hat{O}$ on that space represents the possible *observables* (e.g., position) for the system. If the Hilbert space on which $\hat{O}$ acts has dimension n, then any $\hat{O}$ can be decomposed as the linear sum of at most n mutually orthogonal non-zero projection operators P_k: $\hat{O} = \Sigma_k v_k P_k$, where the v_k are real numbers. (This is guaranteed by the spectral theorem for self-adjoint operators (von Neumann 1932, chap. II.8).) To say that each P_k is a projection operator means that each acts as the identity on every vector in a particular subspace of the Hilbert space—called the *range* of the projection operator—and annihilates (maps to the zero vector) every vector in the Hilbert space orthogonal to its range—called the projection operator's *kernel*: $ker(P_k) = ran(P_k)^{\perp}$, where "⊥" denotes the orthogonal complement. This entails that if the state ψ of a quantum system lies within the range of P_k—$\psi \in ran(P_k)$—then it will satisfy the eigenvalue equation: $\hat{O}\psi = v_k\psi$. In such circumstances, ψ is called the *eigenstate* (or *eigenvector*) of the equation, and v_k its *eigenvalue*. For each state ψ and observable represented by $\hat{O}$, there can be at most one eigenvalue satisfying their eigenvalue equation since the P_k are mutually orthogonal, i.e., their ranges are so. Finally, notice that a projection operator $P$is itself a Hemitian operator, as it can be expressed as $P = 1 \cdot P + 0 \cdot P^{\perp}$, where $P^{\perp}$ is the projection operator such that $ker(P) = ran(P^{\perp})$.[6]

Metaphorically, one can think of an observable as a "question" posed to the system in a certain state with at most n different mutually exclusive and exhaustive answers represented by

[5] The following compressed refresher assumes some familiarity with standard quantum theory. For more extended expositions, see Myrvold (2018, sec. 2), Ismael (2015), and references therein.

[6] At most one of P and $P^{\perp}$ can be the zero operator. In any case, P can be expressed as a real-weighted sum of non-zero projection operators.

the numbers v_k. Each answer affirms a certain property attribution to a system in the corresponding eigenstate. Accordingly, an observable itself can also be thought of as a physical property. For example, the Hamiltonian operator represents the total energy of a system—a property—since its eigenvalues correspond with all the different total energies—also properties—that a system can have. Correlatively, for any observable, each of the projection operators P_k into which it decomposes represents a "yes-no" question asking of a system whether that binary property obtains for the system. For example, the operator representing the spin-x observable for a spin-½ particle such as an electron can be decomposed into projection operators representing the "up?" and "down?" observations. For each of these projection operators, the states lying in its range are those who answer "yes" to the question. (Our discussion of quantum logic in section 2.2 will affirm that those lying in the kernel are those who answer "no.") Thus a property that can take on many values, represented by a self-adjoint operator $\hat{O}$ asking a question with one value per possible answer, can also be represented by the collection of projection operators into which $\hat{O}$ decomposes, each asking a yes-no question.

Promoting this metaphor to reality—reifying the "yes" answers—allows one to represent exactly the properties that a quantum system in a particular state has:

> Eigenstate-Eigenvalue Link (EEL): A quantum system has a (determinate) value v of a property, which is represented by a self-adjoint operator $\hat{O}$, if and only if its state vector is an eigenstate of that operator with eigenvalue v.

In other words, the eigenstates of a self-adjoint operator represent all and only the possible states of a quantum system that have (determinate) values for the property that the operator represents. Famously, this set of possible states is in general not *all* the possible states of the system. Given a self-adjoint operator $\hat{O}$, any state given by a linear sum of eigenstates of $\hat{O}$ (with distinct corresponding eigenvalues), known as a *superposition* of those eigenstates, will fail to have a (determinate) value for the property that $\hat{O}$ represents.

We should pause here to comment on the qualifier 'determinate' that has recently crept into our discussion. Standard formulations of EEL include this qualification (or an analogous one, such as 'definite'), and we are following suit (Myrvold 2018, sec. 2.1; Lewis 2016, p. 76). Its significance is demonstrated in cases where a system is in no eigenstate of a given self-adjoint operator $\hat{O}$ (i.e., when it is in a superposition of $\hat{O}$'s eigenstates). One interpretation of this scenario (consistent with EEL) is that the system simply does not have any value of the property represented by $\hat{O}$. In that case, the qualifier 'determinate' is redundant (and presumably harmlessly so). But other interpretations of this scenario maintain that, *in some sense or another*, it is merely *indeterminate* which value of the property the system has. EEL itself does not fix the exact sense of 'indeterminate' operative here. Only a positive account of quantum indeterminacy can provide that. Accordingly, it is only in the context of discussing any such account that the qualifier 'determinate' takes on any precise meaning (or even logical form). We will have occasion to examine many accounts of quantum indeterminacy below (including our own). But outside of such discussions, we warn the reader that our use of 'determinate' serves mainly as a rather loose placeholder for some more precise concept. In particular, one should certainly not read 'determinate value' as suggesting that, according to EEL, there literally are two types of value (i.e., real number), those that are "determinate" and those that are somehow not.

Let us now return to our general discussion of EEL. One can provide further arguments in favor of EEL. For the "if" direction of the biconditional, one can invoke another postulate of quantum theory, the *Born rule*, which states that if one measures an observable, represented by $\hat{O}$, of a quantum system in a (unit-normalized) state ψ, then the outcome will be one of the eigenvalues v_k with probability $\psi^\dagger P_k \psi$, where $\psi^\dagger$ is the self-adjoint conjugate of ψ and P_k is the (maximal) projector corresponding to the eigenvalue v_k. Importantly, the Born rule connects the formalism of quantum theory with the probabilities of various predictions, not property attribution. But, one can make such a connection by adding the Einstein-Podolsky-Rosen (EPR) "criterion of reality":

> EPR Criterion of Reality: "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to that quantity" (Einstein, Podolsky, and Rosen 1935, 777).

If one interprets the existence of "an element of reality" as the truth of a property ascription, then the "if" direction of EEL follows. For if ψ is in the range of P_k, then $\psi^\dagger P_k \psi = 1$.

Conversely, for the "only if" direction, one can suppose the completeness of quantum theory as a description of reality, and apply the EPR "criterion of completeness":

> EPR Criterion of Completeness: If a physical theory is complete, then "every element of the physical reality must have a counterpart in the physical theory" (Einstein, Podolsky, and Rosen 1935, 777).

From this it follows that every possible property of a physical system has a counterpart in—or is represented by—an element of quantum theory. So if, contra the "only if" direction of EEL, such a system was not in an eigenstate of a given self-adjoint operator $\hat{O}$ yet (determinately) had the property value represented by one of that operator's eigenvalues v_k, then that fact would not be represented in quantum theory, for there are no other interpretive postulates by which it would be represented. This contradicts completeness. Hence it must be the case that no system can have such a property value v_k yet not be in an eigenstate of $\hat{O}$.

Both of these arguments can be—and, in fact, have been—challenged. EPR themselves famously questioned the completeness of quantum mechanics. We shall return to this point in the concluding section as an avenue of exploration for competing accounts of quantum metaphysical indeterminacy. Our goal has not been primarily to defend EEL, but rather show that there is substantial initial plausibility undergirding the orthodox consensus view in support of it. Indeed, although it seems that a philosopher was the first to *coin* EEL (Fine 1973),[7] its conceptual content appears implicitly already in discussions during the 1927 Solvay conference (Bacciagaluppi and Valentini 2009) before increasingly explicit statements and affirmations in even the first modern quantum theory textbooks of the 1930s by Heisenberg (1930), Dirac (1930), and von Neumann (1932), as Gilton (2016) amply documents. By the 1960s it was found in all major textbooks.

2.2 Quantum Logic

EEL already contains the rudiments of an account of indeterminacy for atomic property ascriptions, but to extend it to compound property ascriptions and a propositional logic for indeterminacy requires a brief foray into *quantum logic*, which, though not entailed by EEL, arises naturally from it. Although one can find ample precedent for it in the textbook of von Neumann (1932),[8] his later collaboration with Birkhoff (Birkhoff and von Neumann 1936) fully formalized the semantics for sentences concerning quantum systems as reflecting the structure and geometry of projections in a Hilbert space (or some vector space more generally).[9] In the sequel, we follow the presentation of Dalla Chiara and Giuntini (2002, sec. 1) and de Ronde et al. (n.d., secs. 2–3).[10]

The formalization begins with the correspondence between projection operators and subspaces of a Hilbert space: every projection operator P determines a subspace identical to its range, and every subspace determines a unique projection operator. Because each projection operator represents a yes-no question about a property, by EEL its range therefore represents the (determinate) *extension* of that property—a "yes" answer—for that type of system. In other words, a proposition ascribing that property to a quantum system, expressed by a sentence p, is (determinately) true just in case the system's state ψ lies within the property's (determinate) extension: $\psi \in ran(P)$.[11] Thus, every atomic property ascription has a collection of states as models.[12]

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[7] Fine actually uses the term "eigenvalue-eigenstate link," but the meaning is the same.

[8] In his 1932 textbook, von Neumann focuses only on commuting projection operators P_k and P_j, i.e., ones that satisfy $P_kP_j = P_jP_k$.

[9] There are many traditions associated with quantum logic, some intended as research programs to solve the quantum measurement problem or to characterize quantum theory as non-classical probability theory, and others aimed instead at developing the formal structures of the logic for logicians' purposes. We remain agnostic on all these goals, endorsing quantum logic only as a formalization of property ascription in orthodox quantum theory. See Bacciagaluppi (2009) for a conceptual-historical account of these controversies, which do not bear on our much more limited endorsement.

[10] It will suffice to confine ourselves to a propositional language. The syntax of the language is as usual. For details, including extensions to a first-order language, see, e.g., Dalla Chiara and Giuntini (2002, sec. 2).

[11] For the rest of this section and the next, in which, respectively, we describe quantum logic and then present our own theory of quantum indeterminacy based on that logic, we will largely leave any 'determinate(ly)' qualifiers tacit. That is because, as we are about to see, in quantum logic there is no difference between truth and determinate truth (nor any difference between extension and determinate extension); any indeterminacy amounts to a simple *lack* of either truth or falsity. So there is no harm in leaving out the qualifier in this context, and doing so will simplify our exposition. We will reinstate the qualifiers, as necessary, when we turn to competing theories of quantum indeterminacy in §4.

[12] We will use "property ascription" to refer to both propositional and sentential ascriptions of properties. The context should make clear which is intended on any given occasion.

What about the negation of the property ascription p? The kernel of the projection operator P represents the states that receive a "no" answer to the yes-no question that the operator represents. Hence $\neg p$ is true—and p is false—for a system with state ψ just when $\psi \in ker(P) = ran(P)^{\perp}$. Importantly, the orthogonal complement of $ran(P)$ (i.e., $ker(P)$) always constitutes a *subspace* of the entire Hilbert space. That means it is identical to the range of some (other) projection operator and so, according to EEL, represents the extension of a property. This ensures that $\neg p$, just like p, is a genuine property ascription.

Quantum logic's treatment of negation in terms of ortho-complementation rather than complementation *simpliciter* marks a critical difference between it and classical negation. For while a set and its complement taken together always exhaust the entire space, the same is not true of a subspace and its orthogonal complement: unless $P$is either the identity or the zero operator, $ran(P) \cup ran(P)^{\perp}$is never identical to the entire space. Accordingly, for any such P there will always be states in neither $ran(P)$ nor $ker(P)$—that is, there will always be states for which the property ascription p will be neither true nor false. Thus, we see already that quantum logic's treatment of negation entails that its semantics for quantum property ascription cannot be classical: it violates the principle of bivalence.

A special case of this bivalence failure will be key to our account of indeterminacy below. Consider a self-adjoint operator $\hat{O} = \sum_{k=1}^{n} v_k P_k$ with eigenstates $\varphi_1, \dots, \varphi_n$, and suppose ψ is a unit-normalized linear combination (superposition) of these eigenstates: $\psi = \sum_{k=1}^{n} a_k \varphi_k$, where at least two of the coefficients are nonzero. We noted above that any system in this state will fail to have a (determinate) value for the property represented by $\hat{O}$. This is equivalent to saying that $\psi \notin ran(P_k)$ for each projection operator P_k. But it's also true that for any j such that a_j is nonzero (of which there will be at least two), $\psi \notin ran(P_j)^{\perp}$. Given the semantics of negation, this means that each corresponding p_j is neither true nor false of a system in state ψ. Thus, when a system is in a superposition of the eigenstates of a given operator, there will be at least two properties represented by that operator, ascriptions of which to the system will lack truth values. Conversely, given a quantum state, one can always find a self-adjoint operator in terms of whose eigenstates that state can be expressed as a superposition, and there will still be at least two properties represented by that operator, ascriptions of which to the system will lack truth values. In this sense, superpositions correspond to truth-value gaps in quantum logic.

In contrast with negation, the semantics for the conjunction $p \wedge q$ of property ascriptions p and q is the same as classical logic: $p \wedge q$ is true of a system in state ψ just when $\psi \in ran(P) \cap ran(Q)$, where P and Q are the projection operators whose ranges are the respective extensions of the properties ascribed by p and q. This too is a bona fide property ascription by the lights of EEL, because the intersection of any two subspaces of a Hilbert space is itself a subspace. Accordingly, one can apply the semantics for the negation, above, to determine the states for which $p \wedge q$ is false, namely those in $(ran(P) \cap ran(Q))^{\perp}$.

Disjunctions, however, are quite different. The disjunction $p \vee q$ is true of a system in state ψ just when ψ is in the *span* of the union of the ranges of P and Q, i.e., it can be written as a linear combination (superposition) of elements from those two sets. Unless at least one of P and Q is the identity operator (i.e., the trivial projection), the span of their ranges is a strict superset of their union. But—like with the semantics for the negation—this span is a subspace, and hence represents the extension of a property. One can therefore apply the semantics for the negation, again, to determine the states for which $p \vee q$ is false: those in $(span(ran(P) \cup ran(Q)))^{\perp}$. By contrast, the union of the ranges of non-trivial projection operators P and Q is never a subspace, and hence according to EEL does not represent the extension of any property. This entails that $p \vee q$ is true for some states for which neither p nor q is true.

This last implication has a couple of instances that will be important to our discussion of indeterminacy later on. First, for any p, the disjunction $p \vee \neg p$ comes out true no matter the truth value (true, false, or neither) of p, i.e., $p \vee \neg p$ is true of any state in any Hilbert space. To see this, consider the states for which $p \vee \neg p$ is true: those in $span((ran(P) \cup ran(P)^{\perp})$. It is a mathematical fact, which we won't prove here, that the span of any subspace with its orthogonal complement equals the entire space. It immediately follows that $p \vee \neg p$ is true of any state in the space. Thus quantum logic, despite violating the principle of bivalence, does uphold the law of excluded middle. Second, recall that when ψ is a (non-trivial) linear combination (superposition) of the eigenstates of a self-adjoint operator $\hat{O} = \sum_{k=1}^{n} v_k P_k$, at least two of the property ascriptions p_i, p_j corresponding to projection operators P_i, P_j will lack truth-values. Despite this, the disjunction of all n ascriptions $p_1 \vee ... \vee p_n$ will always come out true. That's because the eigenstates of any operator form a basis for the entire space, which by definition entails that their span—and hence $span(ran(P_1) \cup ... \cup ran(P_n))$—is the entire space.[13]

Both of these examples involve a true disjunction whose disjuncts are neither true nor false. It is important to note, however, that these are special cases. It is not in general true that a disjunction is true whenever both of its disjuncts are neither true nor false. Some such disjunctions will instead be neither true nor false. Thus, the semantics for quantum logic, and in particular that of disjunction, is *non-truth-functional*: the semantic status (true, false, or neither) of a disjunction is not in general determined by the semantic status of its component sentences. Nonetheless, the semantics is still *extensional* in the following sense: both the extension and anti-extension of any complex sentence *p*—i.e., the subspace of states for which p is true, and the subspace of states for which p is false, respectively—are entirely determined by the extensions of its component formulas. This permits us to extend the semantics to sentences of higher complexity by induction on that complexity.

The table below summarizes the semantics for sentences representing property ascriptions.

[13] In fact, the disjunction of just those sentences among $p_1, \ldots, p_n$ that lack a truth value will be true.

Sentence	True for state ψ	False for state ψ
p	$\psi \in ran(P)$	$\psi \in ker(P) = ran(P)^{\perp}$
$\neg p$	$\psi \in ker(P) = ran(P)^{\perp}$	$\psi \in ran(P)$
$p \wedge q$	$\psi \in ran(P) \cap ran(Q)$	$\psi \in (ran(P) \cap ran(Q))^{\perp}$
$p \vee q$	$\psi \in span(ran(P) \cup ran(Q))$	$\psi \in (span(ran(P) \cup ran(Q)))^{\perp}$

Quantum logic has a number of classical features. Letting "⇔" denote logical equivalence and "1" the sentence expressing the trivial proposition—the one whose associated projection operator is the identity operator—we have the following, for sentences p, q, r:

- Commutativity: $(p \wedge q) \Leftrightarrow (q \wedge p)$ and $(p \vee q) \Leftrightarrow (q \vee p)$.
- Associativity: $(p \wedge (q \wedge r)) \Leftrightarrow ((p \wedge q) \wedge r)$ and $(p \vee (q \vee r)) \Leftrightarrow ((p \vee q) \vee r)$.
- Absorption: $(p \wedge (p \vee q)) \Leftrightarrow p$ and $(p \vee (p \wedge q)) \Leftrightarrow p$.
- Excluded Middle: $(p \vee \neg p) \Leftrightarrow 1$.
- Noncontradiction: $\neg(p \wedge \neg p) \Leftrightarrow 1$.
- Involution: $\neg\neg p \Leftrightarrow p$.

Each of these is easily provable.[14] We already remarked on Excluded Middle above. Involution follows immediately from the fact that the operation of orthogonal complementation is its own inverse, so that $ran(P)^{\perp\perp} = ran(P)$ for any projection operator P. For Noncontradiction, notice that no state ψ can be an element of both $ran(P)$and $ran(P)^{\perp}$, as this would require that ψ be orthogonal to itself, and the only element of the space with this property is the zero vector. Thus $ran(P) \cap ran(P)^{\perp}$ is the nullspace, the orthocomplement of which is the entire space. From this it follows that $\neg(p \wedge \neg p)$ is true throughout the entire space, and thus $\neg(p \wedge \neg p) \Leftrightarrow 1$. We leave demonstrations of the rest to the reader.

We've already seen how quantum logic, despite satisfying the principles of excluded middle and noncontradiction, nonetheless violates the law of bivalence. But this failure of bilavence is not its only non-classical feature. Most importantly, the distributive law fails: it is not generally the case that $(p \wedge (q \vee r)) \Leftrightarrow ((p \wedge q) \vee (p \wedge r))$. This shows that quantum logic is not only semantically non-classical (in the sense of violating bivalence), but is also logically non-classical (in the sense that not all classically equivalent formulas will be equivalent in quantum logic).[15] Another deviation from the classical setting is the failure of the semantic version of the deduction theorem for the material conditional: if Γ is a set of sentences ascribing properties to a

[14] They all follow from the fact that the (closed) subspaces of any Hilbert space form an ortho-complemented lattice (Birkhoff and von Neumann 1936).

[15] This marks a critical difference between quantum logic and many-valued Boolean logic. The latter is semantically non-classical but maintains all classical equivalences (Rasiowa and Sikorski 1963).

quantum system, and if every model of $\Gamma \cup \{p\}$ is a model of q, then it is not necessarily the case that every model of Γ is a model of $\neg p \vee q$.[16] Perhaps unsurprisingly, both of these nonclassical features are (partially) owing to the behavior of the disjunction.

3 A Modal Account of Indeterminacy in Quantum Theory

The propositions of quantum logic with truth-value gaps have long been termed "indeterminate" or "indefinite" by the quantum logic and foundations community. Our account of quantum metaphysical indeterminacy takes this way of talking seriously: in order for a property instantiation of a quantum system to be indeterminate, it must be the case that the proposition attributing that property to that particular system is actually neither true nor false, according to EEL and its concomitant quantum logic.

However, this is merely a necessary condition. It is not sufficient because it does not rule out properties that are not assigned a truth value because they are inapplicable to the system in question. For instance, spin-0 quantum systems never have either the spin-up or the spin-down properties in any direction simply because they do not have a positive total spin. So, consider for such a system the question "Is it spin-up or spin-down in the x-direction?", which presupposes that the system is spin-½. Then a sentence attributing to such a system the property of being spin-up in the x-direction would not be true—nor would it be false, for in the context of the question's presupposition this would imply that the system had the spin-down property. However, this truth-value gap seems to have a semantic rather than metaphysical basis because the attribution of a spin direction to such a system most plausibly represents an attributional category error, not a way in which the world is somehow "unsettled." In this case there is not a *propositional* truth-value gap, but merely a *sentential* truth-value gap, owing to that sentence's failure to express any proposition at all in the presupposed context.

To elaborate this issue for present purposes we only need some elementary facts about (weak) superselection sectors, which we draw from Earman (2008, sec. 2), to whose review we refer interested readers for further details. Formally, the issue is that in a slightly more sophisticated version of quantum theory, one models the state space of a quantum system as the direct sum of (say) m different Hilbert spaces H_j: $H_1 \oplus H_2 \oplus \ldots \oplus H_m$. Each element H_j in the direct sum represents a *superselection sector*. The examples above of spin-0 and spin-½ quantum systems represent two such sectors; particle mass and electric charge also can partition a Hilbert space into sectors. Any quantum state resides in exactly one superselection sector;[17] consequently,

[16] Dalla Chiara and Guintini (2002) call this the "Herbrand-Tarski property." The absence of this property for standard quantum logic interestingly parallels the same absence for logics of standard supervaluationist semantics for languages that include a determinacy operator (Fine 1975, 290).

[17] Actually, states that are linear combinations of states in different superselection sectors are possible, but they are mixed states rather than the pure states to which we have confined attention here. We return to the topic of indeterminacy for mixed quantum states in the concluding section.

observables may be defined only over subsets of the sectors.[18] For example, in the case where the state space is the direct sum of Hilbert spaces for spin-0 and spin-½ systems, the spin-up and spin-down observables (relative to any choice of spatial axis determining which directions are "up" and "down") considered in the previous paragraph are defined only on the states in the spin-½ superselection sector. Thus both its range and its kernel are subspaces of this sector. The same holds for any other observable defined only on the states in the spin-½ superselection sector, so any proposition ascribing a property value that can be the outcome of such an observable is neither true nor false for states in the spin-0 sector—similarly for negations, conjunctions, and disjunctions of such propositions. In what follows, we always suppose that when the state space of a quantum system is given, so are its superselection sectors, just as with the aforementioned case of total spin.

In light of this we add the requirement for indeterminate properties that the system to which they are ascribed *could* have the putative indeterminate property, where the scope of the possibility operator at a state—the range of its accessibility relation there—is the state's superselection sector. It's important to restrict the scope of the possibility operator at a state to the states within the actual state's superselection sector for at least two reasons. First, these are the states to which ascription of said property or its negation is not a category mistake. Second, if the scope is unlimited, then even spin-0 states could be spin-½ states (and vice versa), which would undermine the motivation for adding this modal restriction in the first place.

Now, it follows from the first reason just mentioned that the truth of this modal condition scoped within a superselection sector is equivalent to the truth of the excluded middle of a property ascription restricted to that sector. That's to say that, if p ascribes the property in question, then p is possible (in the intended sense) at state ψ if and only if $p \vee \neg p$ is true at ψ. However, we would like to emphasize the explicitly modal version of this condition in our official defining conditions for quantum metaphysical indeterminacy, for it facilitates comparison with other accounts of metaphysical indeterminacy, in §4 below.

> Quantum Metaphysical Indeterminacy: Let S be a quantum system in state ψ. It is *metaphysically indeterminate* whether S instantiates property F if and only if the following two conditions hold:
>
> 1. the proposition <*S* is *F*> is *actually* neither true nor false (according to EEL and standard quantum logic), and
> 2. it's *possible* for <*S* is *F*> to be true.

In the next section we will illustrate our characterization with an example. For now, let us make two remarks to help clarify its content.

First, condition 2 imposes a modal condition on quantum indeterminacy. In this way our account bears a resemblance to other accounts of metaphysical indeterminacy, such as that of

[18] More precisely, they may be defined only over dense subsets of some subspace of the total Hilbert space, but this distinction only makes a difference for infinite-dimensional Hilbert spaces.

Barnes and Williams (2011), that use (something like) a modal operator in the logic of indeterminacy. We will discuss Barnes and Williams's and similar views in §4. For the moment, however, let us emphasize one critical aspect of our view that distinguishes it from these others. As we'll see below, the modal condition that is part of Barnes and Williams's theory (and others) is *global*: if it is indeterminate whether *p*, then there must be a possible world in which *p* is true *and in which every other proposition receives a truth value*—i.e., *p* must be true in a possible world that is classically complete. By contrast, our modal condition is *local*: it requires the possible truth of only *p* itself, saying nothing about the possible truth value of any other proposition (let alone the possibility of jointly assigning truth values to all propositions). This difference will be important for our critical discussion later on (§4.2), since it is the global aspect of Barnes and Williams's view that, we'll argue, makes trouble for their theory.

Second, let us make clear the exact way in which ours is an account of *metaphysical* indeterminacy. Clearly it is not an account of (so-called) *epistemic* indeterminacy. Indeterminacy in our view results in genuine truth-value gaps—situations in which there is *no fact of the matter*—contrary to any epistemic account. The indeterminacy is also not semantic. This is ensured by the fact that our account characterizes such indeterminacy in terms of *propositional* truth-value gaps (as opposed to sentential truth-value gaps). Propositions, unlike sentences, do not admit of imprecision.[19] So wherever the indeterminacy is coming from, it is not coming from an imprecision (or any other feature) of our representations. All that said, we should keep one expectation in check: our account does *not* purport to provide an analysis or reduction of quantum indeterminacy (let alone metaphysical indeterminacy generally). We *characterize* indeterminacy in terms of propositional truth-value gaps, but we are not claiming that this is what such indeterminacy *consists in*. We'll return to this idea below in connection with Torza's account (§4.4).

4 Criticisms and Comparisons

We now turn to a critical discussion of competing accounts of quantum indeterminacy. Appreciation of the limitations of these views will serve to further motivate our positive account just described.

Throughout this section we will work with the following simple test case. Let *e* be an electron and consider just its spin degree of freedom so that its state space will be two-dimensional, one basis thereof being the spin-up and spin-down states in the *x* direction. (In contrast to our discussion of quantum logic in §2.2, where the choice of basis played no important role, such a choice here will facilitate our critical remarks in this section. None of our criticism, however, depends on any of the details of this choice.) Now suppose that *e* is in a superposition of being spin-up ($\uparrow_x$) and being spin-down ($\downarrow_x$) in the *x* direction. We take it as common ground among the alternative theories we will consider below that, if there is any indeterminacy that arises in quantum

[19] At least propositions do not admit of imprecision *in the way that sentences can* (e.g., by semantic indecision). A proposition might be called imprecise if it (precisely) represents an imprecise property or object. But such imprecision, and any indeterminacy it might give rise to, is ultimately rooted in the world, not in any representation thereof.

theory, there is indeterminacy here: it is metaphysically indeterminate whether *e* is $\uparrow_x$ (and likewise indeterminate whether it is $\downarrow_x$).[20] If a theory of indeterminacy cannot successfully model this case, then it is not an adequate model of quantum indeterminacy.

It is only fair, then, that we begin by showing how our theory succeeds on this front. Given that *e* is in a superposition of (the eigenstates corresponding to) being $\uparrow_x$ and being $\downarrow_x$, we know that quantum logic will deliver the result that both of the property ascriptions <*e* is $\uparrow_x$> and <*e* is $\downarrow_x$> are neither true nor false. Each thus satisfies condition 1 of our characterization. Moreover, given that *e* is a spin-½ particle, there is a state within *e*'s state superselection sector such that, were *e* in that state, *e* would be in the range of the projection operator corresponding to the property of being $\uparrow_x$ and so <*e* is $\uparrow_x$> would be true. *Mutandis mutadis* for <*e* is $\downarrow_x$>. Thus both propositions meet condition 2 of our characterization, and so both are indeterminate on our characterization, as desired.

Let's now see how alternative theories fare with respect to our test case. We'll consider four recent theories of metaphysical indeterminacy proposed, respectively, by Calosi and Wilson (2019), Barnes and Williams (2011), Darby and Pickup (2021), and Torza (2020). Our criticism of the first three of these will be quite strong: *given an acceptance of EEL*, each of these theories simply *cannot* model the indeterminacy that arises in quantum theory. So these theories we view as nonstarters (again, given EEL). Our criticism of Torza's view will be more constructive. We think that Torza's theory has many virtues, but also some significant problems. If one fixes these problems in the right way, we argue, one ends up with a view very similar to our own.

We'll consider each of these views in turn, first describing its machinery, then stating its characterization of indeterminacy, then very briefly indicating the motivations for the view, and finally offering our critique. None of our criticisms, we should emphasize, turns on our acceptance of quantum logic. Quantum logic is a part of our own positive theory, but it is not something we assume all other theories will want to embrace. The only substantive assumption we appeal to in our critiques is EEL.

4.1 Calosi and Wilson (CW)

4.1.1 Machinery

CW's characterization is set against a background ontology of two types of entity: states of affairs, in which objects instantiate (or fail to instantiate) properties, and determinable properties (e.g., *being blue*), which (according to CW) are ontologically irreducible to (and just as fundamental as) their determinates (e.g., *being cerulean*). In particular, determinable properties are not reducible to mere disjunctions of determinates. The significance of this irreducibility will be explained

[20] This is clear from Calosi and Wilson (2019), Darby and Pickup (2021), and Torza (2020). Barnes and Williams (2011) never explicitly apply their theory to quantum indeterminacy. But it's reasonable to assume that they agree with the thought that, if there's any quantum indeterminacy, superpositions provide examples of such indeterminacy (Williams 2008b, 765). Finally, as we will see in a moment, Calosi and Wilson probably would object to our description of the relevant indeterminacy as indeterminacy regarding "whether *e* is $\uparrow_x$." But they certainly agree that there is indeterminacy in the present case (if there is any quantum indeterminacy at all).

below.

4.1.2 Characterization

According to CW, indeterminacy is primarily a feature of states of affairs:

> What it is for a state of affairs (SOA) to be indeterminate is for the SOA to constitutively involve an object *o* such that (i) *o* has a determinable property *D* and (ii) *o* does not have a unique determinate of *D*.[21]

Now consider our test case, in which *e* is in a superposition of being $\uparrow_x$ and being $\downarrow_x$. Any indeterminacy that obtains in this case, according to CW, will amount to the existence of an indeterminate state of affairs in the sense just described. Specifically, let SPIN_x be the determinable of *having spin in the x direction*, whose determinates are being $\uparrow_x$ and being $\downarrow_x$., which we will consider as a witness to indeterminacy in our test case. This amounts to the following:

> Our test case is an SOA that constitutively involves *e* such that (i) *e* has the determinable property SPIN_x, and (ii) *e* does not have a unique determinate among being $\uparrow_x$ and being $\downarrow_x$.

CW affirm that the determinable SPIN_x and its determinates, being $\uparrow_x$ and being $\downarrow_x$, are a witness to metaphysical indeterminacy on their view for the quantum system in the sort of state considered in our test case (2019, sec. 4.4)—see also Wilson (2013, 371).[22] We say that this, according to CW, is a "witness" to the indeterminacy in our test case; we don't say, as we might normally, that this captures an indeterminacy in "whether *e* is $\uparrow_x$." This is deliberate. A central aspect of CW's view is that indeterminacy is a property of states of affairs. Accordingly, there is no literal sense in which it could be indeterminate whether an object instantiates a given property. Properties, be they determinates or determinables, are either had or not had—full stop. And propositions ascribing properties are either true or false—full stop:

> MI [metaphysical indeterminacy] does not generate any propositional

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[21] With some harmless simplifications, this characterization is taken verbatim from Wilson's official statement of the view (2013, 366).

[22] Wilson writes of an electron in a superposition state of being $\uparrow_x$ and being $\downarrow_x$ that its state

> is an eigenstate of the operator O corresponding to: Do you have a spin? Hence the system has the property of having a spin. But [its state] is not an eigenstate of the operator O* corresponding to: are you x-spin ↑? So the system does not have the property of having x-spin ↑; nor does it have the property of having x-spin ↓. Here the property corresponding to O acts as the determinable and the property corresponding to O* acts as the determinate, with the system having the determinable *but not any of the corresponding determinates*. (2013, 371; emphasis ours)

That having x-spin ↑ and having x-spin ↓ are the only determinates mentioned of the determinable of interest pragmatically implies that SPIN_x is the determinable represented by O that Wilson has in mind.

> indeterminacy [e.g., in ascriptions of properties], and so no indeterminacy operator [such as 'it is indeterminate whether'] is required. Rather, MI involves a certain pattern of instantiation of determinable and determinate properties; consequently, propositions expressing the obtaining of any given state of affairs (whether precise or imprecise) or the having of any given property (whether determinate or determinable) will, if meaningful, be determinately (i.e., straightforwardly) true or determinately false, as per classical semantic usual. (CW 2019, 2601)

Thus, the indeterminacy involved in our test case arises from *e* simply having the determinable $SPIN_x$ and simply not having any unique determinate among being $\uparrow_x$ and being $\downarrow_x$. There is no literal sense in which it is indeterminate whether it has any of these properties. This feature of CW's view will play a critical role in the first of our critiques below.

4.1.3 Motivation

Wilson (2013) gives a comprehensive defense of CW's general characterization. Let us briefly mention two virtues of the view beyond its supposed extensional adequacy. First, it offers a reductive analysis of the phenomenon of metaphysical indeterminacy. *What it is* for a state of affairs to be indeterminate is just for an individual to instantiate a determinable without instantiating a unique corresponding determinate. CW consider both "gappy" cases, in which no such determinate is instantiated, and "glutty" cases, in which more than one such determinate is instantiated relative to particular circumstances, such as how the different colors of an iridescent feather might be instantiated relative to different physical lines of sight (2019, 2617).[23] If the indeterminacy involved in our test case is gappy, then it's simply not the case that *e* is $\uparrow_x$; nor is it the case that *e* is $\downarrow_x$. It's also not the case that *e* is either $\uparrow_x$ or $\downarrow_x$. None of the corresponding propositions and sentences is true; each is (simply) false. If the indeterminacy in our test case is glutty, then it's the case that *e* is $\uparrow_x$, and it is also the case that *e* is $\downarrow_x$. It's further the case that *e* is both $\uparrow_x$ and $\downarrow_x$. Each of the corresponding propositions and sentences is true, CW maintain, relative to the *x* direction (2019, 2624). Regardless of whether the case at hand is gappy or glutty, *e* has the determinable $SPIN_x$. It's here where CW's insistence on irreducibility secures, for the gappy cases in particular, their account's second virtue, the retention of classical logic. If $SPIN_x$ were reducible to the disjunctive property *being either* $\uparrow_x$ *or* $\downarrow_x$, then *e* couldn't have the determinable $SPIN_x$ and lack both determinates without violation of classical logic. Although this is less of an issue for glutty cases, retaining classical logic in any case makes this account of metaphysical indeterminacy theoretically conservative along the dimension of logic.

[23] CW in fact consider two other sorts of glutty cases: (i) those in which the determinates are not relatively instantiated in the same individual but in different individuals in the same state of affairs; and (ii) those in which the determinates are instantiated to some positive degree, but not fully. We set aside the first sort because CW do not endorse it in any case and the second because CW acknowledge that it would require replacing EEL with a degree-theoretic version (2019, 2621).

4.1.4 Our Critique

We have two independent criticisms of CW's account as well as a further observation of a tension within the account that arises from these criticisms. The first criticism centers on their explicit commitment to classical logic: based on the developments in Wilson (2013), "determinables and determinates provide the basis for an account of metaphysical indeterminacy … compatible with classical logic" (Wilson 2017, §5.3).[24] We maintain that CW's particular characterization of indeterminacy, when applied to our running example of quantum indeterminacy, violates their commitment to classical logic.

To begin, recall that for CW there is no literal sense in which it can be indeterminate whether an object instantiates a given property. It either (simply, determinately) does or it (simply, determinately) doesn't. CW are quite explicit about this. Here is Wilson making the point in the context of (supposed) indeterminate boundaries of macro-sized objects:

> What it is for a macro-object O to have an indeterminate boundary is for it to be *determinately the case (or just plain true)* that (i) O has a determinable boundary property P, and (ii) … O does not have a unique … determinate of P (2013, 373; emphasis ours)

It is clear that, even where there is genuine metaphysical indeterminacy, the relevant object simply and determinately has the relevant determinable and simply and determinately does not have a unique determinate of that determinable.

This rejection of indeterminate property instantiation, moreover, is not just an incidental or freestanding feature of CW's view. It is part and parcel of their particular commitment to classical logic. Contrasting their view with a supervaluationist approach to indeterminacy, for instance, they say:

> On a metaphysical supervaluationist approach … it is possible … to preserve certain theorems of classical logic, though certain classical laws of inference (including contraposition) must be rejected; whether the classical semantic principle of bivalence is preserved depends on whether there is a privileged precisification …. On a determinable-based approach, *and again reflecting that this approach does not generate propositional indeterminacy [e.g., indeterminate property ascriptions]*, no revisions to classical logic or semantics are required (2019, 2601; emphasis ours).

CW's rejection of indeterminate property ascriptions is essential to their commitment to classical logic.

Now consider the proposition <*e* is not $\uparrow_x$>. *Not being* $\uparrow_x$ is a way something can be. So

[24] See also the passage from Calosi and Wilson (2019, 2601) quoted later on in this subsection.

it's a property. Accordingly, by EEL, determinately instantiating this property will correspond to being in an eigenstate of some projection operator. With all this CW seem to agree. Here is Wilson:

> We may think of different quantum operators as asking a given state various questions, such as: Are you spin-up? Are you spin-down? Are you located in the box? Are you alive? And so on. The system is in an eigenstate of the operator if the answer is determinately 'yes' or 'no'. Since the rule says that the system has the property if and only if it is an eigenstate of the relevant operator, then if the answer is not determinately 'yes' or 'no' (equivalently, not an eigenstate), then the system does not have the associated determinate property. (2013, 371)

On the natural semantic assumption that answering "no" to the question "Is *e* spin-up?" is the same as saying that *e* is not spin-up, it's clear that Wilson is here thinking of *not being* $\uparrow_x$ as a property whose determinate instances correspond to an eigenstate of some operator. In other words, *determinately not being* $\uparrow_x$ must correspond to some subspace of *e*'s state space. But by now it should be clear that the qualifier 'determinately' is superfluous in this context. That's because, as we've emphasized, for CW, given their commitment to classical logic, there is no way of instantiating a property other than by determinately instantiating it. Therefore, we can conclude that (simply) *not being* $\uparrow_x$ will correspond with some subspace. But which subspace?

For CW, it must be the subspace corresponding to the property of *being* $\downarrow_x$. That's because CW regard the logic of indeterminacy as extensional, since they view it as classical. So whichever subspace corresponds to the proposition <*e* is not $\uparrow_x$> must be at least implicitly definable from that corresponding to <*e* is $\uparrow_x$>. But there are only four such subspaces: the whole space (always true), the null space (never true), the spin-up space, and the spin-down space. Clearly it can't correspond to any of the first three, so it must correspond to the fourth.

And here lies the problem for CW. Return to our example, in which *e* is in a superposition of being $\uparrow_x$ and being $\downarrow_x$, and so (we are assuming) there is indeterminacy. According to CW's characterization, this implies that *e* does not instantiate a unique determinate among being $\uparrow_x$ and being $\downarrow_x$. Thus, either both of the propositions <*e* is $\uparrow_x$> and <*e* is $\downarrow_x$> are false (if this is a "gappy" case) or both are true (if this is a "glutty" case), relative to the *x* direction. But we just finished arguing that <*e* is $\downarrow_x$> is equivalent to <*e* is not $\uparrow_x$>. Thus, either both <*e* is $\uparrow_x$> and <*e* is not $\uparrow_x$> are false or both are true. In either case, we have a violation of classical logic. At least when applied to the quantum case at hand, CW's characterization of indeterminacy is demonstrably inconsistent with their commitment to classical logic.[25]

[25] In the glutty case, one can demonstrate an inconsistency between EEL and CW's characterization even without appealing to their commitment to classical logic. If our example is a glutty case, *e* instantiates both being $\uparrow_x$ and being $\downarrow_x$. In particular, it follows that *e* is $\uparrow_x$. But recall that for CW, there is no literal sense in which it can be indeterminate whether an object instantiates a given property. It either (simply, determinately) does or it (simply, determinately) doesn't. Therefore we can conclude that in our example, *e* (simply, determinately) is $\uparrow_x$. But from this it follows, by EEL, that *e* is in an eigenstate of the projection operator representing $\uparrow_x$, contradicting the assumption of the example that it is not.

Though the preceding criticism exploits CW's commitment to classical logic, it does not rely on our own endorsement of quantum logic. Nor does our criticism amount to an indictment of applying classical logic to quantum mechanical properties generally (even for those interpretations that accept EEL, such as that described by Glick (2017)). All we are arguing is that CW's particular characterization of indeterminacy cannot be applied to the quantum case without violation of their commitment to classical logic.

We now turn to the second of our two criticisms. Recall that, according to CW, since SPIN_x and its determinates witness the indeterminacy in our test case, *e* fails to instantiate a unique determinate among being $\uparrow_x$ or being $\downarrow_x$, but it nevertheless instantiates the determinable SPIN_x. Now, whatever else determinables are, they are a type of property. Thus SPIN_x is a property, and so by EEL corresponds to some self-adjoint operator. But which operator? It may turn out that there is no unique answer here; maybe SPIN_x can be represented by multiple distinct operators (and we come back to this possibility below). What is important for our purposes, however, is that *one* type of operator that can represent SPIN_x is a *projection* operator. Moreover, any projection operator that does represent SPIN_x must be the *identity operator when restricted to its domain, the spin-½ superselection sector*, which restriction we assume throughout the remainder of this subsection. We are first going to argue for this point. We will then show why this conclusion is problematic for CW's account specifically.

To begin, note that SPIN_x can be represented by some projection operator whose range in turn represents the extension of SPIN_x. That's because SPIN_x is a binary property—a system either has it or does not have it. This just follows from the standard way of thinking about determinables generally. An object is either red or isn't; it's either rectangular or not. Similarly, a system either has a spin in the *x* direction or it doesn't. With all this CW clearly agree, for they write of a system *simply having* determinables like SPIN_x (2019, 2623). This is, of course, in accord with their affirmation that the determinable SPIN_x is a part of the witness to indeterminacy in our test case, as described above in §4.1.2.

That we can represent SPIN_x by a projection operator does not preclude also representing it by some other sort of self-adjoint operator. For example, it might also be natural to represent SPIN_x by the Pauli matrix σ_x, which has two eigenstates, those corresponding to $\uparrow_x$ and $\downarrow_x$, and two eigenvalues, +1 and -1, respectively.[26] In contrast to simply asking the binary question "Do you have spin in the *x* direction?", as a projection operator representing SPIN_x could be thought of, σ_x can be interpreted as asking the non-binary question "Are you $\uparrow_x$ or are you $\downarrow_x$?". We remain neutral on this representational possibility. In any event, if one were to represent SPIN_x by σ_x, this would not preclude also representing it by a projection operator. That's what's important for our present purposes: for CW, SPIN_x is the sort of property that *can* be represented by some projection operator.

The next question is: *which* projection operator? We argue that it must be the identity operator (when restricted to the spin-½ superselection sector). For suppose that SPIN_x were to correspond to some projection operator other than the identity operator. Call this operator *A*. By

[26] In bra-ket notation, $\sigma_x = |\uparrow_x\rangle\langle\uparrow_x| - |\downarrow_x\rangle\langle\downarrow_x|$. For more on this notation, see Ismael (2015, sec. 2).

definition its range cannot contain both the range of the projection operator corresponding to being $\uparrow_x$ and the range of the projection operator corresponding to being $\downarrow_x$, for the span of the union of these two ranges is the whole spin-½ sector, which is uniquely the range of the identity operator. Thus it would always possible for e to be in some state ψ that either is in the range of the operator corresponding to being $\uparrow_x$ or is in the range of the operator corresponding to being $\downarrow_x$, but is *not* in the range of A itself. In other words, it would be possible for e to have one of the determinate properties of being $\uparrow_x$ or being $\downarrow_x$ while at the same time lacking the (supposed) determinable property SPIN_x.[27] But in that case SPIN_x couldn't be the relevant determinable property after all. Whatever else one wants to say about the relationship between determinables and determinates, it's clear that something can't have a determinate property and at the same time lack the corresponding determinable.

We conclude, therefore, that SPIN_x corresponds to the identity operator when restricted to the spin-½ superselection sector. But this conclusion spells trouble for CW's account. To see this, notice that just as we've argued that SPIN_x corresponds to the identity operator on this sector, which is its domain, by similar reasoning one can conclude that for any other direction d, the determinable SPIN_d will likewise correspond to the identity operator on the same sector. On the assumption that EEL reflects a one-to-one correspondence between projection operators and properties, this means that $\text{SPIN}_x = \text{SPIN}_d$ (for any directions x and d), which in turn implies that the determinates of SPIN_d—namely, being $\uparrow_d$ and being $\downarrow_d$—are in fact also determinates of SPIN_x. But this result, which is not problematic in itself, is problematic for CW's account, no matter whether the indeterminacy in our test case is taken to be gappy or glutty. It's problematic if the indeterminacy is gappy, for it implies that there can never *be* any witnesses to gappy indeterminacy (at least for superposition states like the one considered) involving the determinables SPIN_d for any direction d. After all, even if e is in a superposition of being $\uparrow_x$ and being $\downarrow_x$, it will always be in an eigenstate of the spin operator in *some* direction d, and hence will always have one determinate of the determinable SPIN_x (= SPIN_d).[28] Thus no determinable SPIN_d for any direction d is a part of any witness of indeterminacy for our test SOA in the gappy case, contradicting CW's positive claims otherwise (Wilson 2013, 371; Calosi and Wilson 2019, sec. 4.3). Conversely, the result is problematic if the indeterminacy is glutty, because it implies that SPIN_d, relative to any direction d, will *always* be a part of a witness to glutty indeterminacy, no matter the state of e. This is true even when e is in an eigenstate of one of the projection operators corresponding to being $\uparrow_d$ or being $\downarrow_d$, and hence when we should expect no such indeterminacy arising from SPIN_d. This is because there will always be a direction d' according to which the state of e is a superposition of being $\uparrow_{d'}$ and being $\downarrow_{d'}$, and hence e has those two determinates of the determinable $\text{SPIN}_{d'}$ (= SPIN_d).[29]

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[27] Strictly speaking, the fact that the state of e is not in the range of A only allows us to infer that e does not *determinately* possess SPIN_x. But we know that for CW this is no different from saying that e (simply) lacks SPIN_x.

[28] See Wolff (2015) for a detailed discussion of the prospects of treating spin as a determinable. Though Wolff does presume EEL in her essay, she does not consider the problem we outline above.

[29] It may be worth noting that our second criticism would apply just as well were we to take, for instance, *having (non-zero) spin* or *having ½-spin* as the relevant determinable instead of SPIN_x. Consider the former (as similar

In response to this objection, CW may argue that EEL does not, in and of itself, establish the relevant one-to-one correspondence between properties and projection operators. We recognize this fact. But it's worth noting that there is nothing consistent with EEL in the structure of quantum theory that immediately suggests how one could distinguish different properties that have the same (determinate) extension (i.e., correspond to the same projection operator).[30] Not only that, but CW's defense here would require that there be an uncountable number of distinct determinable properties (one for each subspace) that all correspond to the same operator.

This concludes our second critique of CW. No projection operator other than the identity operator (when restricted to the spin-½ superselection sector) could possibly represent the determinable SPIN$_x$ on their account, but this fact, when combined with CW's characterization of indeterminacy, leads to problems adequately capturing the extent of quantum indeterminacy of the sort we (and CW) are concerned with (i.e., that associated with superpositions). If this sort is to be gappy, then witnesses to indeterminacy are maximally undergenerated; if it is to be glutty, then witnesses to indeterminacy are maximally overgenerated. None of this, however, should be taken to mean that the determinable SPIN$_x$ doesn't exist or that (contrary to our previous argument) it doesn't correspond to a projection operator. It just means that CW's particular characterization of indeterminacy in terms of such determinables cannot, given EEL, accommodate the quantum case.

There is one final observation we'd like to make about CW's view. It emerges upon reflection on the two previous criticisms. From the second criticism we know that SPIN$_x$ can be adequately represented by the identity operator when restricted to the spin-½ superselection sector. Now suppose that this sector constitutes the whole Hilbert space. It follows that <*e* is SPIN$_x$> is logically equivalent to any tautology (i.e., any proposition true of the entire space). Given CW's commitment to classical logic, this means that <*e* is SPIN$_x$> will be logically equivalent to <*e* is $\uparrow_x$ or *e* is not $\uparrow_x$> in particular. From the first criticism, however, we know that CW must view *not being* $\uparrow_x$ as logically equivalent to *being* $\downarrow_x$. Putting these ideas together, it follows that, on CW's view, <*e* is SPIN$_x$> is logically equivalent to <*e* is $\uparrow_x$ or *e* $\downarrow_x$>. In other words, the determinable SPIN$_x$ is equivalent to the disjunction of its determinates. But this conflicts with a central part of CW's determinable-based account of indeterminacy:

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reasoning applies to the latter). For reasons that parallel those above regarding SPIN$_x$, the projection operator representing having spin would have to be one which acts as the identity on the spin-½ superselection sector. Therefore, no matter what state *e* is in, it will have the determinable having spin, as expected. However, for any direction *d*, being $\uparrow_d$ and being $\downarrow_d$ will both count as determinates of this determinable. This is also to be expected (since being $\uparrow_d$ is a way of having spin). But now CW face the same exact problems as in the case of SPIN$_x$. Because there is always *some* direction *d* such that *e* is in the eigenstate of one of the operators corresponding to being $\uparrow_d$ and being $\downarrow_d$, it will always have one of the determinates of having spin. It follows that having spin cannot be a part of any witness to gappy indeterminacy. And because there is always some direction *d* such that *e* is in a superposition of being $\uparrow_d$ and $\downarrow_d$, it follows that having spin is always a part of a witness to glutty indeterminacy.

[30] It won't do, for instance, to suppose that SPIN$_d$ and SPIN$_{d'}$, despite having identical extensions, may be distinguishable (when $d \neq d'$) as distinct determinates of the spin-½ determinable referenced in the previous footnote. That's because this supposition is incompatible with a characteristic feature of the relationships between determinables and their determinates as described by Wilson (2017, sec. 2.1): The determinates of a determinable are supposed to be *more specific* ways of being the determinable. But for every direction *d*, any way of being spin-½ is a way of being SPIN$_d$, because the extensions of the two are the same. So in fact being SPIN$_d$ is not a more specific way for something to be spin-½.

> Why look to determinables for insight into MI [metaphysical indeterminacy]? The motivation reflects that determinables are distinctively *unspecific* properties which admit of specification by determinate properties. Other kinds of properties admit of specification: disjunctions are less specific than disjuncts, conjuncts are less specific than conjunctions; a genus is less specific than a species. But nothing prevents these specifiable properties from being themselves precise, or ontologically reducible to precise properties. By way of contrast … determinables are *irreducibly* imprecise, and in particular are not ontologically reducible to any complex combinations of determinates; so they represent a promising basis for characterizing worldly indeterminacy. (2019, 2616–7)

It's clear that, according to CW, determinables are not reducible to mere disjunctions of corresponding determinates. And yet we've just seen how, in certain cases, they must view the determinable SPIN_x as at least logically equivalent to the disjunction of its determinates. Now, we recognize that, in general, there is potential space between logical equivalence and ontological reduction. But in the present context it is not clear that such space exists (or matters). For CW maintain not only that determinables are irreducible, but moreover that they are *irreducibly imprecise* as opposed to merely unspecific. And it is clear from the above passage that disjunctive properties are *not* imprecise in this way; they are merely unspecific. The upshot is that CW must view a property that is "irreducibly imprecise" as logically equivalent to one that is unspecific but perfectly precise. And at this point we must confess that we have lost our grip on what it is to be "irreducibly imprecise."

4.2 Barnes and Williams (BW)

4.2.1 Machinery

BW's theory is set against a background space of ersatz possible worlds—i.e., classically complete, precise representations of reality. For the sake of concreteness, one can think of such a world as a (classically complete) set of (precise) propositions. Now define an *actuality* as any world *w* that *doesn't determinately misrepresent reality*—i.e., it is not the case that it is determinate that *w* misrepresents reality.[31]

4.2.2 Characterization

BW's characterization, applied to the case at hand, goes as follows:

> It's indeterminate whether *e* is $\uparrow_x$ iff there is some actuality in which *e* is $\uparrow_x$ and some actuality in which *e* is not $\uparrow_x$.

[31] The label "actuality" comes from Williams (2008a). In other work BW use different terminology.

It will also be helpful to have their general characterization of *de*terminacy on hand:

> It's determinate that *p* iff *p* holds in every actuality.

4.2.3 Motivation

BW's theory is a type of supervaluationism (broadly construed). It differs from traditional, semantic versions of supervaluationism in that it quantifies over not precise interpretations of an imprecise language but rather precise representations of an imprecise (i.e., indeterminate) reality. Their commonality of form makes BW's theory stand to inherit many of the virtues of traditional supervaluationism. In particular, their theory provides us with a logic and semantics of indeterminacy that preserves the determinacy of both classical theorems and penumbral connections.[32]

4.2.4 Our Critique

The problem with applying BW's theory to the quantum case is that it cannot jointly model four propositions that it ought to be able to:

(a) EEL holds determinately.
(b) Mathematics, and in particular the geometry of Hilbert space, holds determinately.
(c) It's indeterminate whether *e* is $\uparrow_x$.
(d) It's determinate that *e* is $\uparrow_y$.

Suppose that *e* is in an eigenstate of being spin-up in the *y*-direction. Then by EEL, one gets (d). (c) follows from the fact that, being in an eigenstate of spin in the *y*-direction, *e* must be in a superposition of being spin-up and being spin-down in the *x*-direction. (a) is implicit in our overarching assumption of EEL—certainly our assumption is not that EEL is merely *indeterminate*.[33] And (b) is implicit in virtually all theorizing about quantum mechanics. So (a)–(d) should be jointly consistent on any viable theory of quantum indeterminacy (at least, given EEL). And yet they aren't on BW's theory, as we'll now demonstrate.[34]

[32] Penumbral connections are a certain type of necessary connection between concepts or properties, the determinacy of which (one might think) ought to be maintained even in the face of indeterminacy in the instantiation of the individual concepts or properties. For instance, "If Susan and Martha have the same net worth, then Susan is wealthy iff Martha is wealthy" is a penumbral connection that (on this line of thought) should come out determinate even when it's indeterminate whether Susan is wealthy. See Fine (1975) for more on penumbral connections.

[33] One might find it odd to think of EEL as being determinately true, since EEL, being an interpretive principle, has a different status than propositions about the properties of quantum systems. On the other hand, if it weren't determinately true, it would have to be indeterminate or determinately false, and neither of those seems compatible with our assumption of its truth. In any case, in §4.3.4 below we raise an objection to Darby and Pickup's account that does not rely on the determinacy of EEL, and which applies equally to BW's view. See the end of §4.3.4 for relevant discussion.

[34] The following objection also applies to Akiba's (2004) theory, which is structurally similar to BW's.

Suppose each of (a)–(d) is true. From (c) it follows, according to BW's characterization, that there is some actuality *w* at which *e* is $\uparrow_x$. And from (a) and (b) it follows, again according to BW, that both EEL and mathematics hold at every actuality, and hence at *w* in particular. Finally, from (d) it follows that at *w* (and every other actuality) *e* is $\uparrow_y$. But now it is easy to see that *w* is logically inconsistent. At *w*, *e* is both $\uparrow_x$ and $\uparrow_y$. Hence by EEL (which also holds at *w*), *e* is in an eigenstate of the $\uparrow_y$ projection operator *and* is in an eigenstate of the $\uparrow_x$ projection operator.[35] But this means that *e*'s state is the zero vector, which is not a ray in Hilbert space. Since the truths of mathematics also hold at *w*, there is a logical contradiction at *w*.[36]

Our criticism of BW echoes certain others already in the literature. Skow (2010) and Darby (2010) each argue that considerations of the Kochen-Specker Theorem (KST) show that BW cannot model the type of indeterminacy characteristic of quantum mechanics. KST states that for any quantum system whose state space is a Hilbert space of dimension three or greater, if every projection operator on that space corresponds with a (potential) property of the quantum system, and if the system possesses any property independently of how it might be measured—i.e., property assignments are not *contextual*—then not all the system's properties can be assigned values together (Held 2018, sec. 1). Since orthodox quantum mechanics implies both conditions in the antecedent, the conclusion follows for it, too. Along similar lines, Torza (2020) suggests that BW can only model quantum indeterminacy by altering their theory to quantify over *impossible* worlds in addition to possible ones, and he rejects an appeal to such impossibilia on grounds of ontological parsimony.

We don't disagree with the spirit of these criticisms, but we think that our way of formulating the objection does a better job at revealing the underlying weakness of BW's account. It's clear, for instance, that KST is not essential to demonstrating BW's inability to model quantum indeterminacy, as witnessed by our critique above, which never invokes KST. Indeed, KST requires a Hilbert space of dimension at least three, while our counterexample against BW used one of dimension two. KST's force is against a large class of hidden variable theories that already violate EEL by assigning more properties to quantum systems than EEL allows, or by making property assignments contextual in the above sense, which EEL disallows. So if one accepts EEL, then invoking KST against BW is superfluous.

As for Torza's objection to impossible worlds, we agree that BW could, perhaps,

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[35] Technically, in order to infer this conclusion via EEL, we need to assume that, at *w*, it is *determinate* that *e* is both $\uparrow_x$ and $\uparrow_y$. It's easy to establish the second of these in our argument. We simply strengthen our assumption (d) to: (d*) it is determinate that it is determinate that *e* is $\uparrow_y$. This assumption is just as plausible as (d) in our scenario, and for exactly the same reasons. BW's characterization then returns the result that it is determinate that *e* is $\uparrow_y$ in every actuality, and thus at *w* in particular. See Barnes and Williams (2011, §6) for details. It is not as obvious, however, that we are entitled to assume that, at *w*, it is determinate that *e* is $\uparrow_x$. No matter. We still know, for the reasons given above, that *e* is $\uparrow_x$ at *w*. And this at least implies that *e* is not in the range of the operator corresponding to being $\uparrow_y$, which is all that we need to derive a contradiction at *w*.

[36] There's actually a more basic problem for BW. Once we've allowed EEL to be true at all actualities, we're forced to conclude that at *w e* is in an eigenstate of being spin-up in the *x*-direction. But then *w* determinately misrepresents reality after all, since in reality *e* is in an eigenstate of being spin-up in the *y*-direction (and presumably is determinately so). We present the argument in the way we do above because it compares better with similar criticisms in the literature, which we discuss later on.

successfully model quantum indeterminacy if they were to open up their characterization to impossible worlds. But to object to such impossibilia on ontological grounds, as Torza does, is off the mark. There shouldn't be any problem, ontologically speaking, in admitting impossible worlds. After all, these worlds are ersatz—just sets of propositions. If we're happy to allow *possible* worlds, then there's no ground for skepticism about the existence of impossible worlds. The real problem in admitting impossible worlds into BW's theory—and one which Torza himself also seems to recognize later on—is that it undermines the primary virtue of their theory: the retention of classical logic. If BW were to quantify over logically impossible worlds in their characterization of indeterminacy and determinacy, this would result in various contradictions being indeterminately true, which BW decidedly designed their theory to avoid.[37]

4.3 Darby and Pickup (DP)

4.3.1 Machinery

DP's theory is set against a background space of ersatz *situations*: precise, classically consistent, but potentially *partial* representations. Like above, one can imagine situations as sets of propositions, but here we don't require these sets to be complete. Now define a *candidate for representing reality* (or simply "candidate") to be any situation that does not determinately misrepresent reality.

4.3.2 Characterization

DP's characterization, applied to the case at hand, goes as follows:

> It's *indeterminate* whether e is $\uparrow_x$ iff the proposition <e is $\uparrow_x$> is true in some candidate and false in another.

Their general characterization of determinacy modifies BW's in an analogous way:

> It's *determinate* that p iff <p> is true in some candidate and false in none.

4.3.3 Motivation

DP's theory gives us a way of retaining the general supervaluationary flavor of BW's, but now in a way that has the potential to accommodate cases of quantum indeterminacy. To see this, return to the scenario from earlier that gave BW trouble:

(c) It's indeterminate whether e is $\uparrow_x$

[37] Torza really has two objections about the retreat to impossible worlds. The first is the one discussed above concerning matters of ontological parsimony, and which we think misses the mark. The second is very similar to what we claim is the real problem, that it renders certain contradictions indeterminately true. So on this we are in agreement with Torza.

(d) It's determinate that *e* is $\uparrow_y$

On BW's theory, (c) and (d) required that there be some actuality in which *e* was both $\uparrow_x$ and $\uparrow_y$, and this led to trouble. But on DP's theory, (c) and (d) only require that *e* is $\uparrow_x$ in some candidate for representing reality (and is not $\uparrow_x$ in some other such candidate) and that *e* is is $\uparrow_y$ in some candidate for representing reality (and is *not* not $\uparrow_x$ in *any* such candidate). And none of that requires any candidate for representing reality in which *e* is both $\uparrow_x$ and $\uparrow_y$. DP's theory no longer guarantees classical logic, but it still gives us a precise way of modeling the behavior of indeterminacy and determinacy by quantifying over candidate situations.

4.3.4 Our Critique

DP may be able to avoid the particular problem we raised for BW, but that doesn't mean that they can successfully model the scenario under consideration. Recall that the scenario began with the stipulation that *e* is in an eigenstate (with eigenvalue 1) of the projection operator corresponding to being $\uparrow_y$. Of course, there are many states that *e* could be in while being compatible with this stipulation. But call whatever state it is in fact in "ψ". Thus, in addition to (a)–(d), the following is also true of the situation:

(e) It's determinate that *e* is in state ψ.

After all, the fact that *e* is in state ψ is just a stipulation of our example, and there's no reason to think that it would have to (or even could) be indeterminate whether it was in this state. Hence (e). And from (e) it follows that:

(f) It's determinate that *e* is not in any quantum state other than ψ.[38]

Now let *s* be a candidate situation in which *e* is $\uparrow_x$, as guaranteed by (c) on DP's account. What else will be true at *s*? Well, that depends on how one generally understands what it is to be $\uparrow_x$. And here EEL is obviously relevant. To understand what is true at *s*, one can (and should) use EEL. EEL tells us one particular thing that is true at *s*: *e* is in a quantum state ϕ, where ϕ is an eigenstate (with eigenvalue 1) of the projection operator corresponding to being spin-up in the *x* direction.[39]

[38] (e) and (f) are *de re* attributions of indeterminacy, since they are saying, of a particular state, that it is determinate that *e* is in that state (and in no other). This seems (more than) fair to stipulate as part of a scenario for the purposes of our argument. Still, it is worth noting that our objection to DP would go through using only weaker, *de dicto* propositions:

(e*) It's determinate that *e* is in an eigenstate (with eigenvalue 1) of the projection operator corresponding to being spin-up in the *y* direction,

and *mutatis mutandis* for (f).

[39] Just as above (see note 33), to infer this conclusion via EEL, we (arguably) need to suppose that, at *s*, it is *determinate* that *e* is $\uparrow_x$, and it is not obvious what entitles us to this assumption. But as before, this does not affect the

But, given (f) and the fact that $\psi \neq \phi$, that means that *s* determinately misrepresents reality, which contradicts our assumption that *s* is a candidate situation. So DP cannot model the scenario after all.

To close this section, we'd like to make two general comments about our critiques of BW and DP. The first concerns the role of EEL in those critiques. In discussion of BW we noted that EEL is determinate, and hence, on their account, should be true at all actualities. This allowed us to draw a contradiction at an actuality in which *e* was both $\uparrow_x$ and $\uparrow_y$, as required by BW's theory. In our discussion of DP, we did not appeal to the determinacy of EEL. That's not to say that we think it's not determinate; it's just that this fact did not play a role in our critique. Instead, we used EEL as an interpretive principle—which is what EEL is—and *applied* it to our understanding of a particular candidate situation *s*. EEL might not be *true* at *s* on DP's account, even given the determinacy of EEL. But one can (and should) still *apply* EEL to a given situation in our analysis of what is and what is not true at that situation. EEL dictates one aspect of how one understands what it is for a quantum system to be in a particular state and what it is for that system to have a particular property. If one embraces EEL and wants to know what is true at a given situation in which a system has a certain property, there is no reason why one shouldn't apply EEL to that situation.[40] Of course, our argument against DP works just as well against BW (though not vice versa). So really we have *two* distinct arguments against BW. Looking at both types of argument allows us to see the different (though not incompatible) ways of thinking about EEL's role in these models.

The second general comment concerns the phenomenon of so-called "deep indeterminacy," around which much of the recent metaphysics literature on quantum indeterminacy has centered. Deep indeterminacy is indeterminacy that cannot be understood as unsettledness between multiple maximally precise—i.e., classically complete—states of affairs. Skow (2010), who coined the phrase, argues that if there's any quantum indeterminacy at all, it must be deep. One way to appreciate Skow's point is to reflect on the problems we encountered when applying BW's account of metaphysical indeterminacy to the quantum case. BW's characterization requires indeterminacy in *p* to be reflected in a maximally precise world in which *p* and a maximally precise world in which not-*p*. But in the quantum case this is impossible, as we saw above. Given EEL and the mathematics of Hilbert spaces, there is no way to jointly assign values of two properties represented by non-commuting projection operators (e.g., having spin up in the *x*-direction and having spin up in the *y*-direction) to a single system.

Skow's discussion of deep indeterminacy suggests a natural fix to BW's theory: keep the structure of their theory largely in place, but drop the requirement that the relevant worlds be

force of our argument. For we do know that *e* is $\uparrow_x$ at *s*. And from this we can at least infer that *e* is not in an eigenstate of the operator corresponding to being $\uparrow_{y,}$, which is all that we need to derive the contradiction in our argument above.

[40] Here's an analogy. Suppose we embrace counterpart theory (regardless of whether we're modal realists). We are considering a possible world *w* and want to know who, if anyone, at *w* is Saul Kripke. We should apply counterpart theory. That's not because counterpart theory is true *at w*; it's because counterpart theory is part of our background theory that we can and should apply to various worlds in order to know what's true in them.

maximal.[41] DP's theory is an example of such a strategy. However, we just saw that this strategy ultimately fares no better in modeling quantum indeterminacy. The problem is therefore not just the maximality of the states of affairs between which the reality is supposedly unsettled. The problem is rather in characterizing indeterminacy as unsettledness between precise states of affairs at all—whether or not those precise states of affairs are maximal. If we insist on thinking of indeterminacy in terms of multiple "sharpenings," or "resolutions," or "precisifications"—even if these are partial—we won't succeed in modeling quantum indeterminacy (given EEL).

The positive account of indeterminacy we proposed in §3 takes this lesson of deep indeterminacy to heart. That account characterizes indeterminacy directly in terms of truth-value gaps, and makes no attempt to understand it as unsettledness between different precise states of affairs. To appreciate the difference between our view and BW's and DP's, we can contrast the role that modality plays in each. As usual, suppose that it is indeterminate whether *e* is $\uparrow_x$. According to our account, this requires that there be a possible world *w* in which *e* is $\uparrow_x$. But it's easy to see that it also requires that there be a possible world *v* in which *e* is not $\uparrow_x$.[42] So on our account there must be a world in which *e* is $\uparrow_x$ and a world in which *e* is $\downarrow_x$. And this looks a lot like what both BW and DP require. But there are three critical differences.

First, as already mentioned in §3, nowhere do we require that either *w* or *v* is maximal (classically complete). In fact, quite the opposite: neither *w* nor *v* could possibly be maximal, since in each world *e* will be in a state (different from the state it is actually in) that is in neither the range nor the kernel of the projection operator corresponding to being $\uparrow_d$ for any direction *d* not parallel to *x*, and so the proposition <*e* is $\uparrow_d$ > will be neither true nor false at both worlds.

Second, on both DP's and BW's accounts, the existence of worlds *w* and *v* provides necessary and sufficient conditions for its being indeterminate whether *e* is $\uparrow_x$ in reality. But on our account these possibilities satisfy only one of two necessary conditions. The other condition is that <*e* is $\uparrow_x$> be neither true nor false. And it is this condition that is really the central part of our characterization, the modal condition being there merely as a rider intended to rule out spurious cases of indeterminacy, such as truth-value gaps arising from category errors.

Third, and perhaps most importantly, our account does not characterize indeterminacy in reality as unsettledness between *w* and *v*. Instead, the relationship between reality and *w* (and *v*) is that of ordinary counterfactual possibility, and there is no sense in which it is unsettled whether *w* obtains—it simply doesn't obtain. Neither possibility is a sharpening, precisification, or resolution of reality; each is just a different way reality could have been.

All of these features of our view make it better suited, we believe, to accommodate the sort of deep indeterminacy that shows up in quantum theory.

[41] For the remainder of this section, we will understand "possible worlds" in a neutral way, so that, for instance, both BW's complete worlds and DP's partial situations count as possible worlds.

[42] If it is indeterminate whether *e* is $\uparrow_x$, then it will also be indeterminate whether *e* is $\downarrow_x$, which, at least according to quantum logic, is equivalent to saying that it is indeterminate whether *e* is not $\uparrow_x$.

4.4 Torza

4.4.1 Machinery

Torza's theory, like DP's, is set against a background space of ersatz situations. Torza takes these to be (classically consistent, potentially partial) sets of sentences, and we'll follow suit. Now define *the adequate actuality* to be that situation that contains exactly those sentences that are true of reality.

4.4.2 Characterization

Torza's characterization, applied to the case at hand, goes as follows:

> It's indeterminate whether e is $\uparrow_x$ iff neither "e is $\uparrow_x$" nor "e is not $\uparrow_x$" is contained in the adequate actuality.

Determinacy then amounts to:

> It's determinate that p iff $\ulcorner p \urcorner$ is contained in the adequate actuality.

As Torza recognizes, these characterizations reduce to the following:

> It's indeterminate whether e is $\uparrow_x$ iff "e is $\uparrow_x$" is neither true nor false.
>
> It's determinate that p iff $\ulcorner p \urcorner$ is true.

In short, Torza identifies the phenomenon of indeterminacy with that of sentential truth-value gaps.

4.4.3 Motivation

Moving away from supervaluationism and its variants, Torza's theory promises to avoid the problems encountered by BW's and DP's theories. In addition, Torza claims that his characterization constitutes a metaphysical *reduction* of the phenomenon of indeterminacy. This is in contrast to BW's and DP's theories, which provide non-reductive accounts of indeterminacy, owing to their definitions, respectively, of actualities and candidate situations explicitly in terms of determinacy.

4.4.4 Our Critique

As mentioned above, we are generally sympathetic to the spirit of Torza's view. After all, our theory also characterizes indeterminacy in terms of (propositional) truth-value gaps. However, we do have three objections to particulars of Torza's account.

First, Torza's theory does not provide a reductive analysis of indeterminacy, contrary to his advertisement. It's true that his characterization is not circular: there's no invocation of the concepts of determinacy and indeterminacy on its right-hand side. The problem, rather, is that Torza's account proposes to reduce something that is not representational—metaphysical indeterminacy—to something that is representational—sentential truth-value gaps. And this gets things back to front. Whatever metaphysical indeterminacy amounts to, it's not just a matter of certain sentences lacking truth values. On the contrary, sentential truth-value gaps are a *symptom* of metaphysical indeterminacy: "*e* is $\uparrow_x$" is neither true nor false *because* it is indeterminate (in reality) whether *e* is $\uparrow_x$, not the other way around. To reverse the order of explanation here is to confuse the relationship between the world and representations thereof. Our theory is also non-reductive, and for similar reasons. So we are not claiming that our account has an advantage over Torza's on this point. We just think it is important to be clear about what these theories—ours and Torza's—can and cannot be reasonably understood as offering.[43]

Second, Torza's account, as stated, is extensionally inadequate. This is because it counts as indeterminate property ascriptions to quantum systems that are category errors, such as some particular spin direction to a spin-0 system. Our modal condition for indeterminacy, which rules out such spurious cases, distinguishes our account from Torza's.

Third, while Torza's account does avoid the problems particular to BW's and DP's accounts, it also comes at a cost. The latter accounts provide us with a clear way of modeling the behavior of indeterminacy and determinacy—they yield a *logic* of indeterminacy. Torza's account does no such thing—it just tells us indeterminacy results in truth-value gaps. By contrast, our theory, which also relates indeterminacy to truth-value gaps, is built from quantum logic from the get-go.

5 Conclusions and Open Questions

Our account of indeterminacy in orthodox quantum theory is so far the only account that adequately captures this phenomenon and provides a ready-made logic for it. Advocates of theories of metaphysical indeterminacy based on determinables or supervaluationism cannot, contrary to their stated intentions, capture indeterminacy in any orthodox version of quantum mechanics, which accepts EEL. That does not mean that they cannot account for indeterminacy in some non-orthodox version of quantum theory that rejects or modifies EEL. One option is to keep the form of EEL but exchange its property attribution clause for a clause solely about the results of experimental tests. The resulting operational interpretation of quantum logic, in contrast to the realist one we have assumed, dissolves any commitment to representing the real properties of quantum systems with their states (Wilce 2017, sec. 2). Another option is to reject EEL (or at least

[43] Torza (personal correspondence) has recently modified his view to apply to propositions rather than sentences. Assuming propositions are representational, as we understand them in our theory, then this does not make his view any more reductionist than before (just as our account is not reductionist). However, Torza has also suggested to us that he is understanding propositions in a *non*-representational way, e.g., as *facts*. In that case maybe he is providing a genuine reduction after all. Here let us just note one difficulty in moving from propositions (understood representationally) to facts: so-called "truth-value" gaps in facts amount to gaps in the *existence* of facts: no fact that *p* and no fact that not-*p*. And this requires a general ontology of *negative* facts irreducible to the non-existence of positive facts.

its "only if" direction), an option that quite varied realist interpretations take, such as dynamical collapse theories, modal interpretations, and some neo-Everettian interpretations.[44] But this freedom, for either option, comes with the responsibility to articulate just which features of the formalism of quantum theory represent real properties and how. If other accounts of indeterminacy in quantum theory aim to take advantage of this freedom, they have not yet accepted its concomitant responsibility.

That said, two open questions for our own account concern (a) giving a metaphysical interpretation of the logic of indeterminacy and (b) extending the account to other (putative) cases of metaphysical indeterminacy (e.g., vague objects) beyond those that stem from quantum theory. Our discussion immediately following the statement of our conditions for quantum indeterminacy in section 3 suggests some possible answers to both. It could well be that *what it is* for a property of a system, quantum or otherwise, to be indeterminate is that (i) the system is in a state that is a complete description of the system's properties yet neither ascribes nor withholds that property and yet (ii) it is possible for that system to be in a state that ascribes or withholds that property. In the context of quantum theory, superselection sectors provide the adequate scope for the invoked possibility operator; outside of quantum theory it is yet unclear what this scope should be. Above we indicated that condition (ii) for a property of a quantum system is equivalent to the excluded middle holding for that property applied to that system, which avoids explicit invocation of a scoped modal operator, but this equivalence may well be particular to quantum theory, too.

Finally, in addition to these conceptual open questions, there are open technical questions about how to extend the present account to more sophisticated versions of quantum theory that allow for both more complex observables, represented by positive operator-valued measures, and more complex states, namely mixed states. Both cases allow for, roughly speaking, certain convex combinations of observables and states as equally legitimate as the projection-valued measures and pure vector states that we have discussed here. While these combinations are often interpreted as representing a sort of ignorance or epistemic probability distribution over the ontic observables or states, there are substantial problems with such interpretations.[45] Consequently, one might want to adopt an ontological interpretation of these new observables and states that represent a new, intrinsic type of vagueness or unsharpness to states and properties (Busch and Jaeger 2010). Whether such "unsharp quantum reality" is metaphysically tenable, subsumable under the present account, or presents a new form of indeterminacy must await further investigation.

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[44] See Myrvold (2018, sec. 4.2) and Lewis (2016, chap. 4) for brief descriptions of these interpretations and references to further literature, especially the latter for their connection with indeterminacy.
[45] See van Frasseen (1991, chap. 7.3) and references therein for some of the difficulties.

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