# Two Quantum Logics of Indeterminacy

Samuel C. Fletcher and David E. Taylor

**Abstract**: We implement a recent characterization of metaphysical indeterminacy in the context of orthodox quantum theory, developing the syntax and semantics of two propositional logics equipped with determinacy and indeterminacy operators. These logics, which extend a novel semantics for standard quantum logic that accounts for Hilbert spaces with superselection sectors, preserve different desirable features of quantum logic and logics of indeterminacy. In addition to comparing the relative advantages of the two, we also explain how each logic answers Williamson's challenge to any substantive account of (in)determinacy: For any proposition *p*, what could the difference between "*p*" and "it's determinate that *p*" ever amount to?

## 1 Introduction

Is there a sharp cutoff between those people who are tall and those who are not? That is, is there an exact height such that, if you are of that height, you count as tall, but if you are *any* shorter you count as not tall? To many, an affirmative answer to these questions seems wildly implausible: there is no such sharp cutoff between the tall and the not tall. But to answer in the negative is to suppose that for some (possible) people it is simply *indeterminate* whether they are tall—there is *no fact of the matter*. It is the phenomenon of indeterminacy that we will be concerned with in this essay. Borderline cases of vague predicates, such as 'tall', offer paradigmatic examples of indeterminacy. But there are plenty of other (putative) examples across a wide range of subject matter including personal identity (Parfit 1984), reference (Field 1973), translation (Quine 1960), moral obligation (Barnes 2014), causation (Bernstein 2016), fictional entities (Goswick 2019), and future contingents (Barnes and Cameron 2009).

The phenomenon of indeterminacy, though commonplace, is philosophically quite rich. One reason for this is that indeterminacy seems, at least *prima facie*, to clash with the doctrines of classical logic in various ways. Quite generally, for instance, indeterminacy appears to conflict with the law of the excluded middle. If it's genuinely indeterminate whether *p*—if there's really just no fact of the matter—then how can we maintain that, nevertheless, either *p* or not-*p*? Various particular phenomena traditionally associated with indeterminacy also raise their own special puzzles for classical logic. A paradigmatic example of this sort stems from the sorites paradox that is characteristic of vagueness.

Despite these appearances, the dominant conclusion of contemporary work on indeterminacy seems to be that there is, in fact, no tension between indeterminacy and classical logic after all: the logic of indeterminacy just is classical logic. This view has traditionally been associated with theories of *semantic* indeterminacy in particular, i.e., indeterminacy that is merely a product of semantic indecision or some other imprecision in our language or thought

(Fine 1975; Keefe 2000). But the classical view is also now dominant among proponents of *metaphysical* indeterminacy, i.e., indeterminacy that obtains independently of our linguistic or mental representations. Indeed, the three leading theories of metaphysical indeterminacy all subscribe, in different ways, to classical logic (Akiba 2004; Barnes and Williams 2011; Wilson 2013).

Whatever the virtues of retaining classical logic, our purpose in this essay is to motivate and develop a very different sort of approach to the logic of indeterminacy, one centered around a particular type of example. The example we have in mind comes from *quantum mechanics*. Thus suppose an electron *e* is in a superposition of (eigenstates corresponding to) being spin-up and being spin-down in a given direction. A natural (though not mandatory) interpretation of this scenario says that it is genuinely indeterminate whether *e* has the property of being spin-up (or spin-down) in that direction—there is simply no fact of the matter.

The existence of such "quantum indeterminacy," though perennially cited (e.g., Barrett 1999, chaps. 1–2 passim), is not uncontroversial, and the interpretation of superpositions on which it rests is not mandatory. Nevertheless, the interpretive assumptions sufficient to motivate this sort of indeterminacy are relatively meagre. Specifically, quantum indeterminacy would seem to arise on any interpretation of quantum mechanics that subscribes to the *eigenstate-eigenvalue link* (EEL), which defines the exact conditions under which a quantum system has a property. This is significant, given the central role that EEL plays in orthodox quantum theory.[1] Accordingly, it is against the background of orthodox quantum theory, and specifically the assumption of EEL, that our investigation of quantum indeterminacy proceeds in this essay.

The prospect of quantum indeterminacy plays a special role in theorizing about the nature of indeterminacy, since it is widely agreed that *if* there is quantum indeterminacy and EEL is true, then such indeterminacy would have to be *in the world itself*, i.e., an instance of *metaphysical* indeterminacy (Chibeni 2004; Bokulich 2014; P. J. Lewis 2016). This is significant, since many have viewed the very intelligibility or coherence of metaphysical indeterminacy with skepticism (Russell 1923; Dummett 1975; Evans 1978; D. K. Lewis 1986, 212). And while proponents of metaphysical indeterminacy have made an effort to identify a number of examples that seem to exhibit the phenomenon—e.g., among the topics mentioned above, personal identity (Williams 2008), the open future (Barnes and Cameron 2009), and fictional entities (Goswick 2019)—such examples are often metaphysically speculative and contentious. A scientifically grounded example of metaphysical indeterminacy, such as one stemming from quantum mechanics, would go a long way toward defusing skepticism about the phenomenon.

There is, however, another, less commonly appreciated way in which quantum indeterminacy may offer special insight into the workings of indeterminacy generally, and that pertains to the *logic* of indeterminacy. For, examples of quantum indeterminacy appear to exhibit

[1] Not all agree that EEL is orthodox doctrine (Wallace 2012a, 4850; 2012b, 108; 2013, 215; 2019, secs. 6–7). But there is historical evidence of the at least implicit acceptance of this doctrine through analysis of early quantum mechanics textbooks (Gilton 2016), including that of von Neumann (1932), as well as of Einstein at the 1927 Solvay conference (Bacciagaluppi and Valentini 2009, 486).

a logical structure that is quite distinct from that of other examples of indeterminacy (metaphysical or otherwise), such as those associated with vagueness (Darby 2010; Skow 2010). Indeed, we suspect that many of the classical views of indeterminacy mentioned above are primarily motivated by considerations of the logic of vagueness in particular. Focusing on quantum indeterminacy thus offers a new and (we think) fruitful lens through which to consider the logic of (metaphysical) indeterminacy generally.

We take as our starting point the characterization of quantum indeterminacy that we've recently developed in the context of orthodox quantum theory (Fletcher and Taylor 2021). A key component of our theory is our claim that the logic of quantum property ascriptions is *quantum logic*. Now, the semantics of quantum propositional logic are (relatively) well understood (Birkhoff and von Neumann 1936; Dalla Chiara and Giuntini 2002). What is far less understood, however, is how to properly extend these semantics to a language that includes a determinacy (or indeterminacy) operator. Indeed, our recent characterization is only meta-linguistic and does not take the step of introducing a determinacy or indeterminacy operator into the object language. Arguably it is this step that is required in order to provide a quantum logic *of indeterminacy*.[2]

Our aim in this essay is to take this next step. First, we introduce the formalism of elementary quantum theory in §2.1, our own version of standard propositional quantum logic **QL** in §2.2, and our meta-linguistic characterization of quantum indeterminacy in §2.3. Second, in §§3 and 4, respectively, we will present two different ways of introducing operators for determinacy and indeterminacy into **QL**. Each such extension to a quantum logic of indeterminacy preserves a distinct virtue of **QL** with our meta-linguistic account. The first logic preserves the extensionality of **QL**; the second preserves the standard way of interdefining indeterminacy and determinacy, which our meta-linguistic characterization upholds. In general, these logics' two indeterminacy operators turn out to behave in very different manners, owing to the different ways in which each operator interacts with other logical connectives, especially the negation. It is in this sense, then, that we will be presenting two different quantum logics of indeterminacy. Third, as an application of these two logics, we confront in §5 how each affords a distinct response to Timothy Williamson's seminal challenge to the very idea of indeterminacy by rejecting different assumptions of his offered *reductio* (Williamson 1994). Last, in the concluding section §6, we compare the virtues and vices of the two quantum logics, leaving for another occasion the question of which, if either, should be preferred. We also indicate a few other directions for future research, such as extensions to more sophisticated forms of quantum mechanics, logical extensions to a first-order system with identity, and conceptual extensions to other examples of the phenomenon of indeterminacy.

[2] Reichenbach (1944, secs. 30–34) made an early attempt at this sort of project through his three-valued logic for quantum mechanics. Specifically, he introduces "indeterminate" as a third truth value and shows how a sentence of the form "*A* is indeterminate" can be expressed using certain combinations of logical operators applied to the propositional variable *A* (1944, 153). However, his logic is truth-functional and so does not reflect the underlying structure of how subspaces of a quantum system's Hilbert state space, representing the extensions of its possible properties according to EEL, fit together. Quantum logic does just this, but as our examples of the non-truth-functionality of quantum logic in section 2.2.3 demonstrate, to do so it must be non-truth-functional.

One final point about our use of quantum logic before we begin: There are many ways in which quantum logic has been employed in general theorizing about quantum mechanics. It has sometimes been viewed, for instance, as an approach to dissolving the measurement problem or as itself providing an interpretation or reconstruction of quantum mechanics (Wilce 2017, sec. 2; de Ronde, Domenech, and Freytes n.d., sec. 4). We are not advocating for anything like either of these ideas here. All we are doing is adopting the most mundane usage of quantum logic, namely as a formalism that allows us to reason about the properties of quantum systems.

# 2 Quantum Mechanics, Quantum Logic, and Quantum (In)determinacy

## 2.1 Quantum Mechanics

We adopt the standard mathematical framework for elementary quantum mechanics. In particular, we assume that the possible (pure) *physical states* of a given quantum system are represented by unit vectors (or, equivalently, rays) in a finite-dimensional Hilbert space. Each self-adjoint operator Ô acting on a subspace of that space represents a *physical quantity* (e.g., the spin orientation of an electron along a given spatial axis) pertaining to the states in that subspace.[3] Explaining how these operators represent in this way requires certain mathematical preliminaries, which we turn to presently.

First, however, we wish to clarify one point about our talk of self-adjoint operators. It is common to assume that the domain of Ô, $dom(\hat{O})$, is the entire Hilbert space. But we will have occasion to consider exceptions to this when we address superselection sectors below. For that reason, going forward we will typically qualify any discussion of an operator Ô with reference to its domain or, equivalently, the subspace on which it acts. The significance of these qualifications, and our reasons for relaxing the common assumption about the domain of Ô, will become clear in §2.1.3.

### *2.1.1 Mathematical Preliminaries*

We begin with the concept of a(n) (*orthogonal) projection operator*, which will play a central role in what follows. (In the remainder, we focus on these particular projection operators, setting aside the "oblique" projection operators and leaving the qualification "orthogonal" tacit.) A projection operator $P_k$ is a linear operator that is idempotent, i.e., for which $P_k^2 = P_k$. Any such $P_k$ on a given domain can be characterized uniquely by its *range*, $ran(P_k)$, which is the subspace of its domain on which it acts as the identity. For any vector in the domain of $P_k$ that is

[3] We intend this and what follows as a brief synopsis of some well-known formalism for those already familiar with quantum mechanics. See, e.g., Myrvold (2018, sec. 2) and Ismael (2020) for further development and many references to the literature. Our restriction to finite-dimensional Hilbert spaces simplifies many technicalities, but we do not believe that this restriction is essential: any substantive claims we make should be retained under an appropriate extension to the infinite-dimensional case.

orthogonal to $ran(P_k)$, $P_k$ acts as the zero operator. Such vectors form a subspace called the *kernel* of $P_k$, $ker(P_k)$. Thus, $ker(P_k) = ran(P_k)^{\perp}$, where "⊥" denotes the orthogonal complement (in the operator's domain). Finally, to say that *two* projection operators (with the same domain) are orthogonal (to one another) is to say that their ranges are orthogonal, i.e., the range of each is a subspace of the other's kernel.

If the subspace of the Hilbert space on which some self-adjoint operator Ô acts has dimension $n$, then the spectral theorem (for self-adjoint operators) guarantees that Ô can be decomposed as the linear sum of at most $n$ mutually orthogonal non-zero projection operators $P_k$, each with the same domain as Ô. That is, $\hat{O} = \Sigma_k v_k P_k$, where the $v_k$ are real numbers. Conversely, any such linear sum is identical to a self-adjoint operator on that subspace (von Neumann 1932, chap. II.8). This linear sum is called the operator's *spectral decomposition*.

Now consider some state ψ of a quantum system and some self-adjoint operator Ô whose domain includes ψ. ψ will lie in the range of at most one projection operator $P_k$ in the spectral decomposition of Ô. When it does lie in the range of such a projection operator, ψ will satisfy the *eigenvalue equation*: $\hat{O}\psi = v_k P_k \psi = v_k \psi$. In this case, ψ is called the *eigenvector* or *eigenstate* of the equation and $v_k$ its *eigenvalue*.

Finally, note that the sum of mutually orthogonal projection operators with the same domain is a projection operator, and that any projection operator $P$ is itself a self-adjoint operator, for it can be expressed as $P = 1 \cdot P + 0 \cdot P^{\perp}$, where $P^{\perp}$ is the projection operator with the same domain as $P$ such that $ker(P) = ran(P^{\perp})$.[4] Functionally, if $a \in dom(P)$, then $P^{\perp}(a) = a - P(a)$.

*2.1.2 The Eigenstate-Eigenvalue Link*

Given these preliminaries, we can now return to the question of how self-adjoint operators on subspaces of a Hilbert space represent the properties of a quantum system that has that space as its state space. To answer this question we shall, for the purposes of this essay, adopt a standard interpretive assumption of orthodox quantum mechanics, which equates the values of the physical quantities associated to a quantum system with that system's properties:

> Eigenstate-Eigenvalue Link (EEL): A quantum system has a (determinate) value $v$ of a property, which is represented by a self-adjoint operator Ô, if and only if its state vector is an eigenstate of that operator with eigenvalue $v$.

[4] This expression is of the form required by the spectral decomposition because at most one of $P$ and $P^{\perp}$ can be the zero operator on their common domain.

Consider the projection operator equal to the sum of all the projection operators associated with any particular eigenvalue $v$ in a spectral decomposition for Ô. The idea underlying EEL is that the range of this projection operator represents the (determinate) extension of a property: all quantum states in its range (determinately) have the property, and none outside its range (determinately) have the property. Since the projection operators appearing in the spectral decomposition of Ô are orthogonal, a self-adjoint operator represents a collection of mutually exclusive properties, one for each distinct eigenvalue.

For example, consider the spin state of a spin-½ particle such as an electron. The operator represented by the Pauli matrix $\sigma_x$ has two eigenstates within the electron's two-dimensional Hilbert state space, those corresponding to being x-spin up ($\uparrow_x$) and being x-spin down ($\downarrow_x$), and two eigenvalues, +1 and -1, respectively.[5] The respective one-dimensional subspaces spanned by each of these two eigenstates represent the extensions of two mutually exclusive values of the x-spin of the electron. Each of these is itself a property represented by the projection operator on this Hilbert space whose range is this respective subspace. An electron in the x-spin up eigenstate (determinately) has the property of being x-spin up, while one in the x-spin down eigenstate (determinately) has the property of being x-spin down. What about electron spin states that are superpositions of these eigenstates, i.e., those given by a linear sum of these eigenstates? They fall into neither the (determinate) extension of being x-spin up nor the (determinate) extension of being x-spin down. Consequently, by EEL, they fail to have either (determinate) value of the x-spin property. That means that while the (determinate) values of x-spin are mutually exclusive, they are not exhaustive, in the sense that it is not the case that every electron spin state belongs to the extension of one or the other. The analogous claim holds true in general for the values of any other property represented by a self-adjoint operator.

So far, we have not commented on the meaning of the qualifier 'determinate' that appears in EEL and our discussion of it so far, but this discussion of superposition raises it to the fore. Might an electron in a state that is a superposition of being x-spin up and x-spin down nonetheless have an *indeterminate* value of the x-spin property? Might it *indeterminately* have the property of being x-spin up (or x-spin down)? This may seem possible since EEL only proscribes which states have determinate values of properties. However, although it is commonplace to include this 'determinate' qualifier (or a cognate such as 'definite') in discussion of EEL, orthodox quantum mechanics itself does not address what it means for the instantiation of a property, or a value of a property, to be (in)determinate. This must be provided by a supplementary account of quantum (in)determinacy, which would explicate this theoretical term. There are nevertheless some possibilities which can be ruled out immediately. For instance, "determinate value" is not intended to separate the real numbers, from which the eigenvalues are drawn, into two types, the determinate ones and those that are (somehow) indeterminate. Rather, "(in)determinate" should be understood adverbially, pertaining to how, in some way, it is (un)settled which properties obtain. Our own account of quantum indeterminacy in section 2.3

---

[5] In bra-ket notation, $\sigma_x = |\uparrow_x\rangle\langle\uparrow_x| - |\downarrow_x\rangle\langle\downarrow_x|$. For more on this notation, see Ismael (2020, sec. 2).

will explain how this should be understood, but outside of the context of our own account, one should understand reference to 'determinate' values as an undefined term in need of explication.

### *2.1.3 Superselection Sectors*

Recall our assumption about self-adjoint operators from the end of the introduction to §2: they are defined on subspaces of a given Hilbert space rather than only the whole Hilbert space. This assumption prefigures our employment of (weak) *superselection sectors* in our account of quantum indeterminacy, drawing from Earman (2008, sec. 2). Formally, a Hilbert space with superselection sectors is one that decomposes as the direct sum of (say) $m$ different Hilbert spaces: $H_1 \oplus H_2 \oplus \ldots \oplus H_m$. Each element $H_j$ in the direct sum represents a superselection sector, and while there is a single zero vector that belongs to the entire space, any unit vector (pure state) belongs exclusively to one of the sectors or other, thereby precluding (pure) states that are superpositions between states in different sectors.[6]

Historically, prohibitions on certain such superpositions—so-called superselection *rules*—arose from the recognition that certain types of superpositions were never observed, such as of states with different total masses, electrical charges, or integer and half-integer total spin. Adding these rules induces the partitioning of the total Hilbert space into sectors and motivates why certain observables, represented by self-adjoint operators, are only defined on certain subspaces of the total Hilbert space—namely, those subspaces which are superselection sectors. For example, the projection operators associated with being x-spin up and being x-spin down are defined on the superselection sector for being total spin-½, but not on others. Thus, no spin-0 system, for instance, will ever, no matter its state, be in an eigenstate of either of these operators. This corresponds to the intuitive idea that asking such a system whether it is x-spin up or x-spin down is to make a certain type of *category mistake*. This idea, which we will develop further below, will be relevant to our characterization of quantum indeterminacy in §2.3.

## 2.2 Quantum Logic

In the previous section, we introduced the mathematical formalism of elementary orthodox quantum mechanics and specified how that formalism represents the states and properties of quantum systems. But that formalism doesn't by itself provide a complete foundation for understanding the meaning or the truth and falsity conditions of logically complex property ascriptions, such as those involving negations, conjunctions, or disjunctions. In other words, we don't yet have a *logic* of property ascriptions to quantum systems. In what follows, we pursue the idea that the logic of quantum property ascriptions is *quantum* logic. While this approach is not strictly mandatory, it does naturally accompany our assumption of EEL, and it provides the foundation of our characterization of quantum indeterminacy that we have developed elsewhere and which we will describe below (Fletcher and Taylor 2021). Because our concerns here are

[6] Linear combinations of states in different superselection sectors are mixed states rather than the pure states to which we have confined attention here. We return to further considerations regarding mixed quantum states in the concluding section.

primarily semantic, rather than proof-theoretic, we spend the remainder of this subsection developing a semantics for standard quantum logic, one which in particular allows for Hilbert spaces with superselection sectors.

*2.2.1 Syntax*

We adopt a propositional formal language for quantum logic, **QL**, consisting of:

- a countable collection of propositional variables *a, b, c, …*, with or without subscripts;
- two logical operators, the unary negation ¬ and the binary conjunction ∧; and
- parentheses ( ) for punctuation.

We define the formulas for **QL** in the usual recursive way:

- every propositional variable is a formula;
- for every formula α, ¬α is a formula; and
- for every pair of formulas α, β, (α ∧ β) is a formula.

Finally, motivated by de Morgan's law, we introduce the disjunction of two formulas α, β, denoted by (α ∨ β), as an abbreviation for the formula ¬(¬α ∧ ¬β). Because of the controversy around what an adequate conditional might be in quantum logic (Dalla Chiara and Giuntini 2002, sec. 3), we do not include a conditional in **QL**.

*2.2.2 Semantics*

The tradition of standard quantum logic inaugurated by Birkhoff and von Neumann (1936) focused on algebraic semantics for that logic in the guise of complete orthomodular lattices (Wilce 2017, sec. 1.2, 4). The lattice of subspaces of a given Hilbert space is a paradigmatic example of such a structure. However, anticipating our distinct extensions of **QL** in §§3 and 4, we will find it convenient to employ novel semantical structures that directly involve a Hilbert space itself (rather than just a lattice of subspaces) together with a specification of its superselection sectors. This affords us two additional types of structure: subsets of states of a Hilbert space that do not form subspaces, and the special relationship among states of the same superselection sector.

The resulting semantics, which we will describe in the following paragraphs, is closer to the Kripke semantics for quantum logic first proposed by Dishkant (1972) and presented in streamlined fashion by Dalla Chiara and Giuntini (2002, sec. 2).[7] However, our semantics differs from theirs in two important ways (even when we restrict our attention to the common case of a

[7] Dishkant's semantics were actually for a more general class of logics, called orthologics, which at the time were sometimes known as "minimal" quantum logic. As Dalla Chiara and Giuntini (2002, sec. 2) point out, however, the latter name is now misleading as, since that time, even more sparing structures related to quantum logic have been considered and developed.

"Hilbert lattice," i.e., a semantical structure for an orthologic isomorphic to the lattice of subspaces of some Hilbert space). First, Dalla Chiara and Giuntini's semantics employ a valuation function that maps formulas into what are effectively subspaces of the underlying Hilbert space. By contrast, the valuation function we introduce below maps formulas into *projection operators* on subspaces of that Hilbert space. This allows us to encode both the domain and range of the operators as parts of the semantic values of formulas, the former of which we do not assume to be the whole Hilbert space. Second, we assume more structure on the set of nodes ("possible worlds") than do Dalla Chiara and Giuntini and also a different accessibility relation between those nodes. (See footnote 8 for more details on this.)

Let us now turn to the semantics. Define a (finite-dimensional) *Kripke-Hilbert model* of **QL** as a triple $(H, R, V)$ where $H$ is a (finite-dimensional) Hilbert space, $R$ is a binary ("accessibility") relation on the states (unit vectors) of $H$, and $V$ is a function (or "interpretation") from the formulas of **QL** into the set of projection operators defined on one or more superselection sectors of $H$. In addition, we make the following two requirements on $R$ and $V$, respectively.

First, we assume that the accessibility relation $R$ is an equivalence relation that partitions the states of $H$ into superselection sectors.[8] Thus for any two states $\psi$ and $\phi$, $\psi R \phi$ iff $\psi$ and $\phi$ belong to the same (single) superselection sector.

Second, we require that $V$ satisfy the following two conditions for complex formulas:

(i) for any formula $\alpha$, $V(\neg\alpha) = V(\alpha)^{\perp}$, and
(ii) for any pair of formulas $\alpha$, $\beta$,

$$V((\alpha \wedge \beta)) = V(\alpha)_{|dom(V(\alpha)) \cap dom(V(\beta))} \wedge V(\beta)_{|dom(V(\alpha)) \cap dom(V(\beta))},$$

where for any two projection operators $P, Q$ with a common domain, their *meet* $P \wedge Q$ is defined as the projection operator with the same domain and with range $ran(P) \cap ran(Q)$.

[8] Dalla Chiara and Giuntini (2002, sec. 2) instead select the accessibility relation to be reflexive and symmetric but not transitive, so that it holds between any two states that are not orthogonal in the Hilbert space. This allows them to define important structure for a Hilbert space, such as the sets of states that form subspaces, from the accessibility relation, without first assuming that the states arise as unit vectors in a Hilbert space. Indeed, this allows them to characterize the semantics for the more general class of orthologics, not just those that arise from a "Hilbert lattice." However, in the process of doing this they do implicitly assume that the Hilbert space so reconstructed (or the analog thereof for general orthologics) has only a single superselection sector. While this is a common and adequate assumption for many applications, because our characterization of quantum metaphysical indeterminacy refers to Hilbert spaces that may have more than one superselection sector, we must take a different approach. This is why we have allowed ourselves to replace the bare set of nodes ("possible worlds") usually introduced for Kripke semantics with a Hilbert space. That said, it may be possible to proceed with their reconstruction strategy by defining two accessibility relations, one for selection rules (non-orthogonality) and one for superselection rules (same sector), or by defining the latter in terms of the transitive closure of the former. In either case we might use known explicit formulas for the join and meet of two projection operators in terms of their pseudoinverses (Piziak, Odell, and Hahn 1999). We leave this possibility for future investigation.

Condition (i) associates the negation of a formula α to the projection operator with the same domain but that is orthogonal to the one associated with α. Condition (ii) maps a conjunction of two formulas α, β to the projection operator whose domain and range are the intersection of the respective domains and ranges of the operators associated with α, β. These two conditions ensure that the interpretation function $V$ encodes the standard meaning of the negation and conjunction in quantum logic. They also guarantee that for any pair of formulas α, β,

$$V((\alpha \vee \beta)) = V(\alpha)_{|dom(V(\alpha)) \cap dom(V(\beta))} \vee V(\beta)_{|dom(V(\alpha)) \cap dom(V(\beta))},$$

where for any two projection operators $P, Q$ with a common domain, their *join* $P \vee Q$ is defined as the projection operator with the same domain and with range $span(ran(P) \cup ran(Q))$.[9]

Now let $K = (H, R, V)$ be a Kripke-Hilbert model and consider any state $\psi \in H$ and any formula α. We say that α is true of ψ in *K* just when $\psi \in ran(V(\alpha))$ and false of ψ in *K* just when $\psi \in ran(V(\neg\alpha))$. Importantly, α is *neither true nor false* of ψ in *K* when neither of these conditions obtains. Such a truth-value gap is possible in two sorts of situations.

The first involves the sort of superpositions discussed in §2.1.2. Let *P* and *Q* be the projection operators associated with being x-spin up and x-spin down, respectively, and let $V(p) = P$ and $V(q) = Q$. Now suppose *S* is an electron, and that its state ψ is a superposition of being x-spin up and x-spin down. Then both *p* and *q* are neither true nor false of ψ in *K*, even though ψ is in the domain of both *P* and *Q*. This is because the negation of each formula *p* and *q* is assigned an extension identical to that assigned to the other formula.

The second sort of situation in which a truth-value gap arises is one in which the state of a given system is not in the domain of the projection operator corresponding to the property that we are ascribing to that system. For instance, let $V(p)$ and $V(q)$ be as before, but now suppose that *S* is a system in a spin-0 state. Then, like before, neither *p* nor *q* is either true or false of ψ in *K*. But, in contrast with the previous situation, the reason for the lack of truth value in the present case is the simple fact that neither *P* nor *Q* is defined for ψ to begin with, i.e., neither contains ψ in its domain. In this case an ascription of the property *P* (or *Q*) to *S* would seem to involve, or presuppose, a sort of category mistake.

### *2.2.3 Comparison with Classical Logic*

As just discussed, the semantics of **QL** permit that some of its formulas will be neither true nor false of some state ψ in the Hilbert space of a given Kripke-Hilbert model *K*. That is already one significant difference of **QL** with classical logic that has ramifications for its other properties. Yet, those ramifications are not as extreme in every aspect as one might expect from such a change. In the remainder of this subsection, we will discuss some of these: **QL** is not

[9] Interested readers may note an analogy between how the interpretation functions are constrained to act on conjunctions and disjunctions and the truth-value tables for the same in Kleene's weak three-valued logic (Kleene 1952). This is not so surprising considering Kleene's objectives concerning only *partial* recursive functions compared with our own concerning operators defined on only *subspaces* of a given Hilbert space.

truth-functional, but it is still extensional (in a sense); and it maintains many of the logical equivalences of classical logic, including noncontradiction and the excluded middle, but not all: the distributive law fails in general, as does the semantic version of the deduction theorem.

We'll begin with truth functionality, assuming three possible truth values: true, false, or neither. To say that the semantics for a certain logical operator □ are truth-functional is to say that, given a state ψ in the Hilbert space of a given Kripke-Hilbert model *K*, the truth value (among the three) of a formula with □ as its main operator is a function only of the truth value of the formula(s) on which □ operates. The negation, in this sense, *is* truth-functional: for any formula α, ¬α is true/false/neither if and only if α is false/true/neither. What about the conjunction? For any pair of formulas α, β, (α ∧ β) is true when both α, β are true and false when at least one is false (and their projection operators share a common domain).[10] In these cases the truth value of (α ∧ β) is (essentially) a function of the truth values of α, β. But if one of α, β is true and the other is neither true nor false, or if both are neither true nor false, then (α ∧ β) could be either false or neither true nor false. These possibilities can be illustrated in a two-dimensional Hilbert space for an electron's spin in the state of being x-spin up:

- *α is true, β is neither true nor false, and* (α ∧ β) *is false.* Let the projection operator associated with α be that for being x-spin up and let β be that associated with being z-spin down. The projection operator associated with (α ∧ β) is then the zero operator, the orthogonal complement of which is the identity operator, whose domain is the entire Hilbert space.
- *α is true, β is neither true nor false, and* (α ∧ β) *is neither true nor false.* Let the projection operator associated with α be the identity and let β be that associated with being z-spin down. The projection operator associated with (α ∧ β) is then the same as that associated with β.
- *Each of α and β is neither true nor false, and* (α ∧ β) *is false.* Let the projection operator associated with α be that for being z-spin up and let β be that associated with being z-spin down. The projection operator associated with (α ∧ β) is then the zero operator.
- *Each of α and β is neither true nor false, and* (α ∧ β) *is neither true nor false.* Let the projection operator associated with each of α and β be that for being z-spin up. The projection operator associated with (α ∧ β) is then the same as that associated with α and β.

[10] If both α and β are true, then the state ψ lies in the domain and range of each of the projection operators associated with α and β, hence lies in the range of the projection operator with their common domain and common range. If at least one of α or β is false, then the state ψ lies in the domain of the associated projection operator—say, that for α—and in the range of its orthogonal complement. That guarantees that it will never lie in the common range of that operator and the one associated with the other—say, that for β. If that latter operator has the same domain as the former, then by definition the intersection of their ranges will be a subset of that for α, hence the orthogonal complement of that intersection will be a superset of that for α.

Each of these examples uses projection operators with a common domain on a single superselection sector. So, the same range of possibilities obtains with projection operators with different domains or with more than one superselection sector. Moreover, because the disjunction is defined from the conjunction and the negation, the same sorts of failures of truth-functionality arise for it as for the conjunction.

Nevertheless, **QL** *is* extensional in a straightforward sense. Say that the extension of a formula $\alpha$ is the set of states of which $\alpha$ is true, and the anti-extension of $\alpha$ is the set of states of which $\alpha$ is false. Then **QL** is extensional in the sense that for each logical operator, the extension of each formula with that operator as its main operator is a function of the extension and anti-extension of the formulas on which it operates. One can see this immediately from conditions (i) and (ii) in the definition of *V*, which provide the extension of each such formula explicitly in terms of the extensions and anti-extensions of its immediate subformulas. (The orthogonal complement, domain, and range of a projection operator are all definable in these terms.)

We now move on to compare the logical equivalences of **QL** and classical logic. To facilitate our discussion, we introduce the following metalinguistic symbology. Let "$\Leftrightarrow$" denote the binary relation of logical equivalence between two formulas $\alpha$ and $\beta$, i.e., the relation that holds between $\alpha$ and $\beta$ when, for all Kripke-Hilbert models *K* and all states $\psi$ in the Hilbert space of *K*, $\alpha$ is true/false/neither of $\psi$ in *K* if and only if and $\beta$ is true/false/neither of $\psi$ in *K*. Further, let "$1_\alpha$" denote a formula such that the projection operator $V(1_\alpha)$ acts as the identity operator on domain $dom(V(\alpha))$. Then, for any formulas $\alpha, \beta, \gamma$, we have the following:[11]

- Involution: $\neg\neg\alpha \Leftrightarrow \alpha$.
- Commutativity: $(\alpha \wedge \beta) \Leftrightarrow (\beta \wedge \alpha)$ and $(\alpha \vee \beta) \Leftrightarrow (\beta \vee \alpha)$.
- Associativity: $(\alpha \wedge (\beta \wedge \gamma)) \Leftrightarrow ((\alpha \wedge \beta) \wedge \gamma)$ and $(\alpha \vee (\beta \vee \gamma)) \Leftrightarrow ((\alpha \vee \beta) \vee \gamma)$.
- Absorption: $(\alpha \wedge (\alpha \vee \beta)) \Leftrightarrow (\alpha \wedge 1_\beta)$ and $(\alpha \vee (\alpha \wedge \beta)) \Leftrightarrow (\alpha \wedge 1_\beta)$.[12]
- Noncontradiction: $\neg(\alpha \wedge \neg\alpha) \Leftrightarrow 1_\alpha$.
- Excluded Middle: $(\alpha \vee \neg\alpha) \Leftrightarrow 1_\alpha$.

Involution follows from the fact that the orthogonal complement is its own inverse, i.e., $ran(P)^{\perp\perp} = ran(P)$ for any projection operator *P*. Commutativity, associativity, and absorption follow from the same properties of the meet and join operations applied to any pair of projection operators. Noncontradiction follows from the mathematical fact that for any projection operator $P$, $ran(P) \cap ker(P)$ is the zero vector, hence $P \wedge P^{\perp}$ is the zero operator on $dom(P)$ and

[11] We adapt this list and the following comparison with the distributive law from de Ronde et al. (n.d., sec. 2).

[12] Those familiar with Boolean algebra may recognize that the right-hand side of these equivalences are more complicated than the usual absorption laws. The conjunction with $1_\beta$ is necessary, however, for if $V(\alpha)$ and $V(\beta)$ do not share the same domain, then $V((\alpha \wedge \beta))$ and $V((\alpha \vee \beta))$ are defined only on the intersection of the domains of $V(\alpha)$ and $V(\beta)$. If $V(\alpha)$ and $V(\beta)$ do share the same domain, then $V((\alpha \wedge 1_\beta)) = V(\alpha)$.

$(P \wedge P^{\perp})^{\perp}$ is the identity operator on $dom(P)$. Excluded middle then follows from the definition of the disjunction, together with involution, commutativity, and noncontradiction.

There are, of course, differences between **QL** and classical logic. A paradigmatic such difference concerns the distributive laws, $(\alpha \wedge (\beta \vee \gamma)) \Leftrightarrow ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$ and $(\alpha \vee (\beta \wedge \gamma)) \Leftrightarrow ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$, which *do not* hold in **QL** in general. To see this, consider, as before, a two-dimensional Hilbert space for an electron's spin. Let $\alpha, \beta, \gamma$ be mapped to the projection operators representing being z-spin up, x-spin up, and x-spin down, respectively. Then $V((\alpha \wedge (\beta \vee \gamma))) = V(\alpha)$, but $V(((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)))$ is the zero operator. And $V((\alpha \vee (\beta \wedge \gamma)))$ is the zero operator, but $V(((\alpha \vee \beta) \wedge (\alpha \vee \gamma)))$ is the identity operator. The failure of the distribution laws shows one critical way in which **QL** is distinct from the class of many-valued Boolean logics, which preserve all classical logical equivalences (Rasiowa and Sikorski 1963).

Another deviation from the classical setting is the failure of the semantic version of the deduction theorem for the material conditional: let Γ to be a set of formulas such that, whenever every element of the set $\Gamma \cup \{\alpha\}$ is true of some ψ in a Kripke-Hilbert model *K*, β is also true of ψ in *K*; even so, it is not necessarily the case that whenever every element of the set Γ is true of ψ in *K*, $(\neg\alpha \vee \beta)$ is also true of ψ in *K*. This can be expressed succinctly by the claim that even though $\Gamma \cup \{\alpha\} \vDash \beta$, it may not be the case that $\Gamma \vDash (\neg\alpha \vee \beta)$, where we read "⊨" as "entails." To see this, consider a three-dimensional Hilbert space *H*. Let $\Gamma = \{\}$ and let *K* be a Kripke-Hilbert model such that *V*(*p*) is some projection operator whose domain is *H* and whose range is a two-dimensional subspace, and *V*(*q*) is some projection operator whose range is a one-dimensional subspace of the range of $V(p)$. Then whenever $p$ is true of some ψ in *K*, $q$ is true of ψ in *K*—indeed, this is so for all states ψ in *K*. But it's not the case that $(\neg p \vee q)$ is true of all ψ in *K*, for $ran(V(\neg p \vee q))$ is only a two-dimensional subspace of *H*.

What is the source of these deviations from classical logic? Famously, Birkhoff and von Neumann (1936, 837) declared that

> The main difference seems to be that whereas logicians have usually assumed that properties ... of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the *distributive identities* ... as the weakest link in the algebra of logic.

The "properties of negation" to which they refer are involution, noncontradiction, excluded middle, and a version of contraposition in which the conditional involved expresses that its antecedent's extension is a subspace of its consequent's extension These are indeed all valid logical equivalences in **QL**. Nevertheless, there is still an important sense in which it *is* the negation which is responsible for the non-classical features of **QL**. One can see this already in the semantics for the conjunction. For, even though a conjunction's truth conditions are the same as in classical logic, its falsity conditions make special reference to the negation: any formula α is false of a state ψ in a Kripke-Hilbert model *K* if and only if ¬α is true of ψ in *K*. And while according to classical Kripke semantics this condition would be equivalent to the usual condition that α is false of ψ in *K* just in case $\psi \notin V(\alpha)$, this is not so in **QL** owing to the semantics for the negation. The gap between these is precisely the gap into which formulas fall when they are neither true nor false of ψ in *K*. In §§3 and 4, this observation will be an important part of how we develop our two extensions to **QL** involving determinacy and indeterminacy operators.

## 2.3 Quantum (In)determinacy

We now turn our attention to the topic of indeterminacy in quantum mechanics. Because our focus in this section is on the metaphysics (broadly construed) of quantum property instantiation, rather than on the formal semantics for **QL** presented above, our discussion will be centered around the status of the *propositions expressed* by (interpreted) formulas, rather than the formulas themselves. The relationship between the two is as one would expect: Let $K = (H, R, V)$ be a Kripke-Hilbert model, where *H* is a Hilbert space representing the possible states of a system *S*. Then for any formula α, the *proposition expressed by* α is <*S* is $F_\alpha$>, where being $F_\alpha$ is the physical property represented by the projection operator $V(\alpha)$, as per EEL.

### *2.3.1 Metalinguistic Characterization*

In this subsection we describe the characterization of quantum indeterminacy that we've developed in previous work. We begin with the intuitive association of indeterminacy with truth-value gaps: where there are truth-value gaps (at least of a certain type), there would seem to be indeterminacy. Recall from §2.2.1 that there are two distinct ways in which truth-value gaps can arise in our semantic theory. Let us consider each in turn, describing matters in terms of propositions, rather than (interpreted) formulas.

The first source of truth-value gaps can be described in terms of superpositions. If an electron *S* is in a superposition of being x-spin up and x-spin down, then the proposition <*S* is x-spin up> is neither true nor false (and similarly for <*S* is x-spin down>). In this straightforward sense, then, there is simply *no fact of the matter* whether *S* has the property of being x-spin up, and so we have a case of genuine indeterminacy. Moreover, the existence of this indeterminacy is not in any way owing to how we represent its subject matter. Whether the physical system *S* instantiates the physical property of being x-spin up is not something that depends on how we, in either thought or language, represent either that system or that property. Thus, if it is *indeterminate* whether *S* instantiates being x-spin up, this indeterminacy must be entirely in the world itself. This vindicates the popular thought that, if there is quantum indeterminacy, that indeterminacy must be *metaphysical* in nature.

The second source of truth-value gaps involves situations in which a system is in a state for which the projection operator corresponding to the property being applied to that system is simply not defined. For instance, if we suppose *S* is a spin-½ particle such as an electron, then the proposition <*S* is x-spin 0> will be neither true nor false, no matter the state *S* is in. Mathematically, this is represented by the fact that the state vector of *S* will always lie inside the spin-½ superselection sector and thus outside of the domain of the projection operator corresponding to x-spin 0. Physically, this is owing to the fact that *S*, being a spin-½ particle, is just not the sort of thing that can possibly be x-spin 0 in the first place, since being x-spin 0 is a property exclusively of systems with integer total spin.

Unlike the first type of truth-value gap, this second type is not, to our minds, a genuine instance of indeterminacy. Rather, it is akin to a *category mistake*. To call an electron x-spin 0 is to do something analogous to what you do when you call the number two blue. But saying that it is *indeterminate* whether two is blue seems to be a misapplication of the concept (Taylor 2018). The reason for this, in both cases, is that it just doesn't make any sense for the electron to be x-spin 0 or for two to be blue in the first place. In any case, even if we do want to recognize these as cases of indeterminacy, they clearly are not cases of *metaphysical* indeterminacy. For the indeterminacy here, such as it is, is owing simply to the fact that we are misapplying a *predicate* 'x-spin 0' or 'blue', to something for which it is simply not defined. There's no indeterminacy in the world here; just a misapplication of language. Finally, it is worth emphasizing that there is nothing about this second type of truth-value gap that is in any way distinctive to quantum mechanics, since category mistakes show up across virtually all subject matter and corresponding parts of language.

For all these reasons, our focus in what follows will be exclusively on the sort of quantum indeterminacy exhibited by the first type of truth-value gap, and thus which is metaphysical in nature. This is in keeping with the general consensus that, if there is any especially interesting indeterminacy that arises out of quantum mechanics, that indeterminacy is metaphysical in nature and paradigmatic examples of such indeterminacy are those associated with superpositions. From this perspective, the second source of truth-value gaps, arising from category mistakes, represents a spurious case of the sort of indeterminacy we are concerned with, and should be ruled out by any characterization thereof.

We are now in a position to state our characterizations of determinacy and indeterminacy. These characterizations are based on both EEL and quantum logic as recounted above. We begin with the determinacy:

> Quantum Metaphysical Determinacy (QMD): Let *S* be a quantum system in state ψ. It is *metaphysically determinate that* S instantiates property *F* if and only if the proposition <*S* is *F*> is *actually* true.

Our characterization of QMD draws from both EEL and our semantics from quantum logic involving Kripke-Hilbert models. Since the proposition <*S* is *F*> is a property ascription, EEL provides the conditions under which it holds determinately, namely those in which the state of the system ψ is an eigenstate of the projection operator representing the property *F*. Our semantics for quantum logic then tell us that these are precisely the conditions in which <*S* is *F*> is true *simpliciter* with respect to ψ. Finally, the indexical "actually" in QMD just makes explicit that the state ψ with respect to which the proposition is being evaluated is in fact the actual state of the system. Putting these ideas together, QMD tells us that for <*S* is *F*> to be determinate just is for it to be (actually) true (simpliciter).

Now here is our characterization of indeterminacy:

> Quantum Metaphysical Indeterminacy (QMI): Let *S* be a quantum system in state ψ. It is *metaphysically indeterminate whether S* instantiates property *F* if and only if the following two conditions hold:
>
> 1. the proposition <*S* is *F*> is *actually* neither true nor false, and
> 2. it's *possible* for <*S* is *F*> to be true.

Condition 1 captures the idea that quantum indeterminacy involves truth-value gaps, which we have witnessed above. It draws on QMD, EEL, and quantum logic. By QMD and EEL, the proposition <*S* is *F*> is actually true just in case ψ is an eigenstate of the projection operator representing *F*. Quantum logic affirms this, adding the conditions under which the proposition is false, hence fixing the conditions under which it is neither true nor false. It is exactly these latter conditions that Condition 1 requires to be satisfied in order for it to be indeterminate whether *S* instantiates *F*.

Condition 2 is designed to exclude the spurious instances of (metaphysical) indeterminacy associated with the second type of truth-value gap, arising from category mistakes, that we saw above. It too draws on QMD, EEL, and our Kripke-Hilbert semantics. To say that it is possible for <*S* is *F*> to be true is to say that there is a possible state ϕ of *S* such that <*S* is *F*> is true when evaluated for ϕ. And ϕ is a possible state for *S* just in case there is some (unitary or projective) dynamical process which would eventually transform its state ψ into ϕ. Such dynamical processes can connect all and only the states in the same superselection sector. So, it is possible for <*S* is *F*> to be true of ψ just in case there is some state ϕ such that ϕ is in the same superselection sector as ψand ϕ is an eigenstate of the projection operator corresponding to the property *F*. (In the formal semantics, this scope of possibility is represented by the accessibility relation of a Kripke-Hilbert model, which partitions the underlying Hilbert space into its superselection sectors.) By contrast, category mistakes involve property attributions that would always be inapplicable: <*S* is *F*> is a category mistake when there is no dynamical process that would transform the state of *S* into an eigenstate of the projection operator corresponding to the property *F*.

There's an equivalent characterization of Condition 2 worth noting here. It takes advantage of the fact that the law of the excluded middle is valid in **QL**, as discussed in §2.2.3. If we take the formula *f* of **QL** to express the proposition <*S* is *F*> according to an interpretation $V$ from a Kripke-Hilbert model *K*, then since $(f \lor \neg f) \Leftrightarrow 1_f$ and $dom(V(1_f)) = dom(V(f))$, which is the superselection sector(s) on which $V(f)$ is defined, it's *possible* for <*S* is *F*> to be true for ψ in *K* if and only if $(f \lor \neg f)$ is true of ψ in *K*. This equivalence will play an important role in the next subsection and in sections 3 and 4, as it will allow us to formalize Condition 2 without introducing a modal operator into the formal language.

To close this subsection, we'd like to emphasize that Condition 2 in our characterization of QMI is one of our main innovations, and makes it more extensionally adequate than other accounts of QMI, such as that of Torza (2020), that include only some variation on the first condition.[13] For instance, take again the example we have been considering throughout, the spin state of an electron. Supposing it to be in a superposition state ψ of being x-spin up and x-spin down, then it is metaphysically indeterminate whether the electron is, say, x-spin up because <the electron is x-spin up> is neither true nor false of ψ, yet <the electron is x-spin up or the electron is not x-spin up> is true of ψ. But whether the electron has x-spin 0 is not metaphysically indeterminate because even though <the electron is x-spin 0> is neither true nor false of ψ, <the electron is x-spin 0 or the electron is not x-spin 0> is also neither true nor false—condition 2 is not satisfied—because being x-spin 0 is a property of systems with integer total spin, while the electron has total spin ½.

### *2.3.2 Formal Characterization as an Extension of **QL***

Our characterization of (quantum metaphysical) determinacy and indeterminacy so far is only in the meta-language: "being (in)determinate that" is not an operator implemented in **QL**. It is, however, elementary to add operators for determinacy and indeterminacy whose semantics mirror our meta-language definitions. We do so as follows. First, we add the unary operators Δ and ▽, representing "it is determinate that" and "it is indeterminate whether," respectively, to the language.[14] Second, for any formula α of **QL**, we let Δα and ▽α be "determinacy formulas," a special extra class of formula-like expressions. Third, add the truth-value conditions for determinacy formulas as follows:

Let *K* be a Kripke-Hilbert model with interpretation function *V*. Then:

- Δα is true of ψ in *K* if and only if α is true of ψ in *K*; otherwise, Δα is false.
- ▽α is true of ψ in *K* if and only if (i) neither α nor ¬α is true of ψ in *K* and (ii) (α ∨ ¬α) is true of ψ in *K*; otherwise, ▽α is false.

Note that conditions (i) and (ii) in the semantics for ▽α hold if and only if $\psi \in dom(V(\alpha)) \setminus (ran(V(\alpha)) \cup ran(V(\alpha)^{\perp}))$, which is in general not a subspace of the Hilbert space. In other words, the extension of ▽α is not in general a subspace of the Hilbert space; it may be merely a subset. This fact will play a critical role in sections 3 and 4.

[13] There are also other differences with Torza's account. For instance, his version of condition 1 is in terms of sentential truth-value gaps, rather than propositional ones. See Fletcher and Taylor (2021, sec. 4.4) for further discussion and critique of this position.
[14] As suggested by their difference in grammar, these operators are not logical duals of one another. Instead, if Δ is understood as analogous to necessity, then ▽ would be understood as analogous to contingency (rather than possibility).

Introducing determinacy and indeterminacy operators in this way is fine as far as it goes. But it doesn't go very far, since it only provides a semantics for a select class of new expressions, namely those of the form $\Delta\alpha$ and $\nabla\alpha$, where $\alpha$ is a formula of the original language. As a result, this approach faces at least two important and connected deficiencies, one conceptual and one logical. Conceptually, the current approach only allows one to make claims about first-order (in)determinacy, and yet the question of whether there is a viable, non-trivial account of higher-order indeterminacy is an open philosophical question (Greenough 2003). One should not foreclose on an answer to this question just for the sake of convenience. Logically, there is little motivation to separate determinacy formulas as a special class of formula-like expressions, as the current approach does. Such expressions should be included in the recursive definition of the formulas of the formal language.

There is good reason, then, to try to move beyond the limitations of the current approach—in particular, to extend the original language to include $\Delta\alpha$ and $\nabla\alpha$ in the recursive definition of formulas. While there is no *syntactic* obstacle to this extension, it does raise important questions about the semantics for complex formulas with (what we have called) determinacy formulas as subformulas. In particular, the semantics for **QL** presented in §2.2 assumes that each formula of the language is associated with a projection operator on a subspace of the Kripke-Hilbert model's Hilbert space, one whose range is the extension of the proposition that the formula expresses. But this assumption is false for an expression like $\nabla\alpha$, whose extension is not guaranteed to be a subspace of the Hilbert space, as noted above. As a result, our original semantics for negations and conjunctions will not be well defined for formulas involving $\nabla\alpha$, for those semantics generally presuppose that the formulas on which these connectives act are associated with projection operators.

In each of §§3 and 4, we present a distinct extension to the semantics of **QL** to overcome these problems. The central difference between these two extensions concerns how each treats the negation. Specifically, each provides a different answer to the following, motivating question: How should we extend our original semantics for $\neg\alpha$ given that, in the extended language, the extension of $\alpha$ is no longer guaranteed to be a subspace of the underlying Hilbert space?

# 3 Quantum Logic of Indeterminacy, Extensional Version

The first extension of **QL** answers this motivating question roughly as follows: we extend the semantics for the negation to act as the set complement for the extensions of formulas that are not subspaces—i.e., those not represented by projection operators—while still acting as the orthocomplement for extensions of formulas that are subspaces. Since this piecewise semantics for the negation depends only on the extension of the immediate subformula on which the negation acts, we call this the *extensional* version of the quantum logic of indeterminacy, $\mathbf{QLI_E}$.

## 3.1 Syntax

We extend the language of **QL** with two additional symbols, the unary operators Δ and ∇, representing "it is determinate that" and "it is indeterminate whether." We then define the formulas for $\mathbf{QLI_E}$ in the same recursive way as we did for **QL**:

- every propositional variable is a formula;
- for every formula α, ¬α, Δα, and ∇α are each formulas; and
- for every pair of formulas α, β, (α ∧ β) is a formula.

As before, motivated by De Morgan's laws, we introduce the disjunction of two formulas α, β, denoted by (α ∨ β), only as an abbreviation for the formula ¬(¬α ∧ ¬β).

## 3.2 Semantics

In our original semantics for **QL**, each formula α was assigned a projection operator as its semantic value. The extension of α was given by the range of that operator, which is by definition a subspace of the underlying Hilbert space. In our extended language, however, there are certain formulas to which our semantics should assign extensions that are not subspaces, but mere subsets, of the Hilbert space. The semantic values of such formulas therefore cannot be projection operators. To remedy this issue, we first introduce the concept of a *set-projection operator*. Set-projection operators act similarly to projection operators but their ranges need not be subspaces of the Hilbert space. Officially, we can define this notion as follows:

> Let *H* be a Hilbert space and $A \subseteq H$ be a subspace of *H*. Then we say that a function $P: A \rightarrow A$ is a *set-projection operator* on *A* when, for all $a \in A$, it can be written in the form $P(a) = a\chi_B(a)$, where $B \subseteq A$ is a union of one-dimensional subspaces of *H* and $\chi_B$ is the characteristic function for *B*, which equals 1 on elements of *B* and 0 otherwise.

Clearly, for any set-projection operator $P$, $ran(P) = B$ and $ker(P) = \{0\} \cup (A \setminus B)$, and specifying both of these sets, or one of them and $dom(P)$, uniquely determines $P$, just as with projection operators. And set-projection operators are also idempotent, just like projection operators. But unlike projection operators, they are not linear unless they are trivial—i.e., the zero or identity function.

We now define a *generalized projection operator* as any operator that is either a projection operator or a set-projection operator. It is these operators that will play the role of semantic values in our extended language below. On the generalized projection operators—i.e., the projection and set-projection operators—we can define operations analogous to those for projection operators only, such as the orthocomplement. For any generalized projection operator $P: A \rightarrow A$, define its *complement* $P': A \rightarrow A$ as $P'(a) = a - a\chi_{ran(P)}(a)$. It follows that

$P'(a) = a\chi_{A\setminus ran(P)}(a)$. Thus, for any generalized projection operator $P$, $P'$ is the set-projection operator on the domain $A$ with $ker(P') = ran(P)$. We can also extend the definition of the meet $P \wedge Q$ for two projection operators (given in §2.2.2) to two generalized projection operators $P, Q: A \to A$, at least one of which is a set-projection operator, as the set-projection operator with domain $A$ and range $ran(P) \cap ran(Q)$.

Now define a (finite-dimensional) *Kripke-Hilbert model* of $\mathbf{QLI_E}$ as a triple (*H, R, V*) where *H* is a (finite-dimensional) Hilbert space, *R* is the "accessibility" equivalence relation on the states (unit vectors) of *H* whose equivalence classes are the superselection sectors, and *V* is a function (or "interpretation") from the formulas of $\mathbf{QLI_E}$ into the set of generalized projection operators defined on one or more superselection sectors of *H*, which satisfies the following five conditions:

(i) if α is a propositional variable, then $V(\alpha)$ is a projection operator;
(ii) for any formula α,

$$V(\neg\alpha) = \begin{cases} V(\alpha)^{\perp}, & V(\alpha) \text{ is a projection operator,} \\ V(\alpha)', & \text{otherwise.} \end{cases}$$

(iii) for any pair of formulas α, β,

$$V((\alpha \wedge \beta)) = V(\alpha)_{|dom(V(\alpha))\cap dom(V(\beta))} \wedge V(\beta)_{|dom(V(\alpha))\cap dom(V(\beta))};$$

(iv) for any formula α, $V(\Delta\alpha) = V(\alpha)$; and
(v) for any formula α, $V(\nabla\alpha) = V(\alpha)' \wedge V(\neg\alpha)'$.

Let us make a few remarks regarding this definition of *V*. First, conditions (i)–(iii) ensure that each Kripke-Hilbert model for $\mathbf{QLI_E}$ is an extension of one for **QL**: they agree on the interpretation of all formulas that $\mathbf{QLI_E}$ and **QL** share in common. (Note that the third condition uses the definition of the meet for generalized projection operators.) Second, the piecewise semantics for the negation stated in condition (ii) marks a critical difference between $\mathbf{QLI_E}$ and **QL**: it distinguishes the action of the negation on formulas represented by projection operators from those represented by set-projection operators. Third, conditions (iv) and (v) provide the semantics for the determinacy and indeterminacy operators, respectively. They both encode directly our meta-linguistic characterization of determinacy: determinacy picks out states for which a proposition is (simply) true, and indeterminacy picks out states for which a proposition is neither true nor false, yet still possible. Finally, in light of condition (i), it is only through condition (v) that any set-projection operators can be assigned to formulas. In particular, when $V(\alpha)$ is neither the identity nor zero operator on its domain for a formula α, then $V(\nabla\alpha)$ will be a set-projection operator.

To say more about how condition (v) adequately captures our meta-linguistic account of indeterminacy, we must complete the semantics for $\mathbf{QLI_E}$ by providing the truth conditions of formulas for particular states within a given Kripke-Hilbert model $K = (H, R, V)$. In fact, they are the same in form as those for **QL**: for any state $\psi \in H$ and any formula α, α is true of ψ in *K* just when $\psi \in ran(V(\alpha))$, and α is false of ψ in *K* just when $\psi \in ran(V(\neg\alpha))$. As with **QL**, α is neither true nor false of ψ in *K* when neither of these conditions obtains.

In our discussion of **QL**, we identified two distinct ways in which a formula α can exhibit such a truth-value gap. One of these involves a sort of category mistake: the operator $V(\alpha)$ is simply not defined for the state ψ of the system, i.e., $\psi \notin dom(V(\alpha))$. The possibility of this sort of truth-value gap carries over to $\mathbf{QLI_E}$ in full generality: for any $\mathbf{QLI_E}$ formula α, unless $V(\alpha)$ is defined on the entire Hilbert space, there will always be some states lying outside of its domain and thus for which α will be neither true nor false (in *K*).

The other source of truth-value gap that we identified in **QL** is associated with situations in which the state ψ of the system is a superposition of the eigenstates of $V(\alpha)$, i.e., $\psi \in dom(V(\alpha))$ but $\psi \notin ran(V(\alpha)) \cup ran(V(\alpha))^{\perp}$. This type of truth-value gap is of much greater theoretical significance, since, unlike gaps arising from category mistakes, it represents a type of genuine metaphysical indeterminacy. However, and importantly, this type of truth-value gap does not carry over to $\mathbf{QLI_E}$ in full generality; it only applies to formulas interpreted as projection operators. More precisely, if ψ is a superposition of the eigenstates of $V(\alpha)$, then α will be neither true nor false for ψ (in *K*) if *and only if* $V(\alpha)$ is a projection operator. By contrast, if $V(\alpha)$ is *not* a projection operator—i.e., if it is a set-projection operator—α will always be either true or false of ψ in *K*, because the truth and falsity conditions for such formulas are exhaustive: $V(\neg\alpha) = V(\alpha)'$, and so $\psi \in ran(V(\neg\alpha))$ if and only if $\psi \notin ran(V(\alpha))$ (again, under the supposition that ψ is in a superposition of the eigenstates of $V(\alpha)$ and thus in the superselection sector(s) of $V(\alpha)$ to begin with).

To close this subsection, we'd like to demonstrate how the $\mathbf{QLI_E}$ semantics for ∇α do indeed capture our meta-linguistic characterization of indeterminacy for property ascriptions from §2.3. Recall our characterization's two necessary and jointly sufficient conditions for it to be indeterminate whether a system *S* instantiates property *F*: <*S* is *F*> must actually be neither true nor false while being possibly true. Now suppose that we translate the property ascription <*S* is *F*> as the propositional variable *p* in $\mathbf{QLI_E}$, and let (*H, R, V*) be an arbitrary Kripke-Hilbert model for $\mathbf{QLI_E}$. Then, for any state ψ, we have that:

$\nabla p$ is true for ψ in $K$ iff $\psi \in ran(V(\nabla p))$

iff $\psi \in ran(V(p)' \wedge V(\neg p)')$

iff $\psi \in ran(V(p)' \wedge V(p)^{\perp\prime})$

iff $\psi \in ran(V(p)')$ and $\psi \in ran((V(p)^{\perp})')$

iff (i) $\psi \notin ran(V(p))$ and $\psi \notin ran(V(p)^{\perp})$, and (ii) $\psi \in dom(V(p))$
iff (i) $p$ is neither true nor false for ψ, and (ii) $(p \lor \neg p)$ is true for ψ
iff (i) <*S* is *F*> is neither true nor false for ψ, and (ii) it is possible that <*S* is *F*> is true.

Finally, supposing that ψ is the actual state of system *S*, then we have that:

$\nabla p$ is true for ψin *K* iff (i) <*S* is *F*> is actually neither true nor false, and (ii) it is possible that <*S* is *F*> is true.

This verifies that our truth conditions for $\nabla p$ coincide with our meta-linguistic characterization of indeterminacy.

## 3.3 Some Logical Equivalences, Entailments, and Other Logical Properties

All but one of the logical equivalences for **QL** considered in §2.2.3 hold in $\mathbf{QLI_E}$ as well.[15] The one exception is one of the two forms of absorption: $(\alpha \lor (\alpha \land \beta)) \Leftrightarrow (\alpha \land 1_{\beta})$. It turns out this equivalence fails in $\mathbf{QLI_E}$ whenever $V((\alpha \land \beta))$ is a projection operator but $V(\alpha)$ is not. The source of this failure is the way in which the definition of the disjunction interacts with the semantics for the negation in just this case. To illustrate, let $V$ be an interpretation function on some Kripke-Hilbert model, and suppose that $V(\gamma)$ is a projection operator but $V(\alpha)$ is not. Then we have that:

$$V((\alpha \lor \gamma)) = V(\neg(\neg\alpha \land \neg\gamma)) = (V(\alpha)' \land V(\gamma)^{\perp})^{*},$$

where the * operation denotes the orthocomplement $\perp$ if the generalized projection operator it acts upon is a projection operator; otherwise, it denotes the complement $'$. Now suppose that the Hilbert space *H* of the Kripke-Hilbert model is two-dimensional, with basis $\{a, b\}$, and let $V(p) = P_a$ and $V(q) = P_{a+b}$ for propositional variables $p, q$, where $P_x$ is the projection operator with domain *H* and range given by the span of the vector denoted by *x*. It follows that $V((\nabla p \land q)) = V(q)$ is a projection operator. So, letting $\alpha = \nabla p$ and $\gamma = (\nabla p \land q)$ in the above inset equation, we have that

$$V((\nabla p \lor (\nabla p \land q))) = (V(\nabla p)' \land V((\nabla p \land q))^{\perp})^{*} = (V(\nabla p)' \land V(q)^{\perp})^{*} = V(1_p),$$

[15] The proofs for these are generally analogous to their proofs in **QL**. The only complication is for disjunctive associativity when not all of the formulas are either property ascriptions or non-property ascriptions. In each of these cases, one can apply the inset formula below, or a variation on it, to expand each of the two formulas in question.

which is not equal to $V(\nabla p)$.

What about logical equivalences and entailments involving the determinacy and indeterminacy operators? In the following, let α be a formula of $\mathbf{QLI_E}$ and suppose that a Kripke-Hilbert model has been given with interpretation function $V$.

1. $(\Delta\alpha \vee \neg\Delta\alpha) \Leftrightarrow 1_{\alpha}$. This is an application of the excluded middle with the formula Δα.
2. $\Delta\alpha \Leftrightarrow \alpha$. This follows immediately from the constraints on the interpretation function for any Kripke-Hilbert model.
3. $\Delta^{n}\alpha \Leftrightarrow \Delta^{m}\alpha$ for non-negative integers $n, m$, where for positive $n$, $\Delta^{n}$ abbreviates $n$ concatenated copies of Δ, and $\Delta^{0}\alpha$ just denotes α. This follows from the previous logical equivalence by application of mathematical induction.
4. $\neg\Delta\neg\alpha \Leftrightarrow \alpha$, since $V(\neg\Delta\neg\alpha) = V(\Delta\neg\alpha)^{*} = V(\neg\alpha)^{*} = V(\alpha)^{**} = V(\alpha)$.
5. $\neg\alpha \Leftrightarrow \neg\Delta\alpha$, since $V(\neg\Delta\alpha) = V(\Delta\alpha)^{*} = V(\alpha)^{*} = V(\neg\alpha)$.
6. $\Delta\neg\alpha \Leftrightarrow \neg\Delta\alpha$, since $V(\Delta\neg\alpha) = V(\neg\alpha) = V(\alpha)^{*} = V(\Delta\alpha)^{*} = V(\neg\Delta\alpha)$.
7. $\neg\nabla\nabla\alpha \Leftrightarrow 1_{\alpha}$. This follows by applying constraint (v) for Kripke-Hilbert models, the law of excluded middle, and the observation that $V(\nabla\alpha)$ is never a projection operator on its domain unless it is the zero or identity operator.

Equivalences (3) and (7) bear upon the question of higher-order indeterminacy for $\mathbf{QLI_E}$. According to the first, if a proposition holds determinately, then it is determinate that it does so, and vice versa; according to the second, it is never the case that a proposition asserting indeterminacy is itself indeterminate. While is possible that ∇Δα be true, this is not clear evidence of higher-order indeterminacy since equivalence (2) indicates that this expresses the same proposition as ∇α. In other words, the semantics for the determinacy operator in $\mathbf{QLI_E}$ make it so trivial that it could be dispensed with in the formal language; it is only the semantics for the indeterminacy operator which provides new expressive power over **QL**.

The slim connection between determinacy and indeterminacy can be captured in the following entailments, none of whose converses hold.

8. $\Delta\alpha \vDash \neg\nabla\alpha$. This follows since
   $ran(V(\Delta\alpha)) = ran(V(\alpha)) \subseteq ran((V(\alpha)' \wedge V(\neg\alpha)')^{*}) = ran(V(\neg\nabla\alpha))$.
9. $\neg\Delta\alpha \wedge \neg\Delta\neg\alpha \vDash \nabla\alpha$. The left-hand side is actually a contradiction, by equivalences (4) and (5).

The fact that the converse of entailment (9) does not hold shows that $\mathbf{QLI_E}$ does not admit the usual characterization of indeterminacy as bearing the same sort of relationship that contingency does to necessity. Indeed, indeterminacy is not in general definable in terms of determinacy because the extension of $\nabla\alpha$ can be a subset yet not a subspace of the underlying Hilbert space. By contrast, any formula that applies a negation, conjunction, or determinacy operator to subformulas interpreted as projection operators will always itself be interpreted as a projection operator; in other words, its extension is always a subspace.

# 4 Quantum Logic of Indeterminacy, Intensional Version

The extensional version of our quantum logic of indeterminacy, $\mathbf{QLI_E}$, distinguished between two types of formula and corresponding proposition: those whose extensions are subspaces of the underlying Hilbert space and those whose extensions are mere subsets of that space. The intensional version of our quantum logic of indeterminacy, which we will denote $\mathbf{QLI_I}$ and expound presently, also draws a distinction between two types of formula (and corresponding proposition), but it instead draws that distinction at the syntactic level.

## 4.1 Syntax

As with $\mathbf{QLI_E}$, we include the unary operator $\Delta$, representing "it is determinate that", in the vocabulary of $\mathbf{QLI_I}$. We also define the formulas for $\mathbf{QLI_I}$ in the same recursive way:

- every propositional variable is a formula;
- for every formula $\alpha$, $\neg\alpha$ and $\Delta\alpha$ are each formulas; and
- for every pair of formulas $\alpha, \beta$, $(\alpha \wedge \beta)$ is a formula.

As for the unary operator $\nabla$ representing "it is indeterminate whether," we only introduce it as an abbreviation: for any formula $\alpha$, '$\nabla\alpha$' abbreviates $(\neg\Delta\alpha \wedge \neg\Delta\neg\alpha)$, the denial that any one of $\alpha$ and $\neg\alpha$ is determinate. This is similar to how we introduce the disjunction of two formulas $\alpha, \beta$, denoted by $(\alpha \vee \beta)$, as an abbreviation for the formula $\neg(\neg\alpha \wedge \neg\beta)$, motivated by De Morgan's laws.

In addition to the formulas, we also define syntactically a separate class of expressions, the *property ascriptions* of $\mathbf{QLI_I}$, as the formulas it shares with **QL**. Previously, we used the phrase "property ascription" in an informal sense to denote the type of proposition that is the intended interpretation of **QL**. This new formal definition codifies this designation. The property ascriptions thus also admit of a recursive definition:

- every propositional variable is a property ascription;
- for every property ascription $\alpha$, $\neg\alpha$ is a property ascription; and
- for every pair of property ascriptions $\alpha, \beta$, $(\alpha \wedge \beta)$ is a property ascription.

Any formula of $\mathbf{QLI_I}$ that is not a property ascription will be called a *non-property ascription*; these consist in all the formulas containing a determinacy operator, $\Delta$.

The distinction between these two types of formulas, the property ascriptions and non-property ascriptions, will play an important role in the semantics of $\mathbf{QLI_I}$, and has a plausible justification in reference to that and in connection with EEL: a claim about (in)determinacy is not a claim indicating that a system possesses a certain property, but a claim about such a fact. So, there are not some properties of quantum systems that, contrary to EEL, were not properly represented by some projection operator within the context of standard quantum logic. That's also reflected in the original, meta-linguistic account of quantum metaphysical indeterminacy in section 2.3.

## 4.2 Semantics

The semantics for $\mathbf{QLI_I}$ employ the same basic machinery as the semantics for $\mathbf{QLI_E}$ given in §3.2, namely generalized projection operators and the operations of meet and complement defined on such operators.

Define a (finite-dimensional) *Kripke-Hilbert model* of $\mathbf{QLI_I}$ as a triple (*H, R, V*) where *H* is a (finite-dimensional) Hilbert space, *R* is the "accessibility" equivalence relation on the states (unit vectors) of *H* whose equivalence classes are the superselection sectors, and *V* is a function (or "interpretation") from the formulas of $\mathbf{QLI_I}$ into the set of generalized projection operators defined on one or more superselection sectors of *H*, which satisfies the following *five* conditions:

(i) if α is a propositional variable, then $V(\alpha)$ is a projection operator;

(ii) for any property ascription α, $V(\neg\alpha) = V(\alpha)^{\perp}$;

(iii) for any non-property ascription α, $V(\neg\alpha) = V(\alpha)'$;

(iv) for any pair of formulas α, β,

$$V((\alpha \wedge \beta)) = V(\alpha)_{|dom(V(\alpha))\cap dom(V(\beta))} \wedge V(\beta)_{|dom(V(\alpha))\cap dom(V(\beta))}; \text{ and}$$

(v) for any formula α, $V(\Delta\alpha) = V(\alpha)$.

Conditions (i), (ii), and (iv) ensure that the interpretation a Kripke-Hilbert model provides of the property ascriptions—the formulas that $\mathbf{QLI_I}$ shares with **QL**—is the same as that which a Kripke-Hilbert model of **QL** would provide. (When condition (iv) is applied to a pair of formulas at least one of which is a non-property ascription, the result is the join of their respective generalized projection operators.) Condition (iii) crucially distinguishes the semantics for the negations of non-property ascriptions from those of property ascriptions: the extensions of the former are given by complements instead of orthocomplements.[16] This will entail, when we

[16] Semantically, the determinacy operator formally implements in the object language something close to what Randall and Foulis (1983, 843) call the "canonical map," which embeds the lattice of subspaces into the lattice of subsets of states. However, they interpret the subspaces as operationally testable propositions and the subsets as property ascriptions. See also Foulis et al. (1983, sec. 4) for more details on how this map is constructed in their framework.

come to the truth conditions for formulas below, that non-property ascriptions are always either true or false; they do not witness any truth-value gap.

Condition (v) provides the interpretation of the determinacy operator, and hence also of the indeterminacy operator. Like with $\mathbf{QLI_E}$, a formula of $\mathbf{QLI_I}$ with $\Delta$ as its main logical operator receives the same interpretation as its immediate subformula, reflecting our characterization of determinacy in §2.3 as amounting simply to truth. But unlike $\mathbf{QLI_E}$, for $\mathbf{QLI_I}$ a formula with $\nabla$ as its main logical operator receives an interpretation more directly in line with the relationship between determinacy and indeterminacy found in standard characterizations of indeterminacy (namely one in which $\nabla\alpha$ is logically equivalent to $(\neg\Delta\alpha \wedge \neg\Delta\neg\alpha)$).

We can now state the truth conditions of formulas for particular states within a given Kripke-Hilbert model $K = (H, R, V)$. These conditions are the same as they were for both $\mathbf{QLI_E}$ and **QL**: for any state $\psi \in H$ and any formula $\alpha$, $\alpha$ is true of $\psi$ in *K* just when $\psi \in ran(V(\alpha))$, and $\alpha$ is false of $\psi$ in *K* just when $\psi \in ran(V(\neg\alpha))$. As with **QL** and $\mathbf{QLI_E}$ before, $\alpha$ is neither true nor false of $\psi$ in *K* when neither of these conditions obtains.

The types of situations in which a formula may exhibit a truth-value gap in $\mathbf{QLI_I}$ are also similar to those in which it may in **QL** and $\mathbf{QLI_E}$, though some qualifications are in order. Formulas expressing category mistakes receive the same treatment in $\mathbf{QLI_I}$ as before: when $V(\alpha)$ is simply not defined for the state $\psi$ of the system—i.e., when $\psi \notin dom(V(\alpha))$—the formula $\alpha$ is neither true nor false for $\psi$. On the other hand, in the situation in which $\psi$ is a superposition of the eigenstates of $V(\alpha)$, $\alpha$ will be neither true nor false of $\psi$ if *and only if* $\alpha$ is a (non-trivial) *property ascription* (in the technical sense defined in §4.2). The possibility of this sort of truth-value gap follows from the fact that, as discussed in §4.2, the property ascriptions of $\mathbf{QLI_I}$ are just those formulas it shares with **QL**, which are interpreted by Kripke-Hilbert models in the same way. By contrast, if $\alpha$ is a non-property ascription, it will always be either true or false of $\psi$ in *K*, because the truth and falsity conditions for such formulas are exhaustive: $V(\neg\alpha) = V(\alpha)'$, and so $\psi \in ran(V(\neg\alpha))$ if and only if $\psi \notin ran(V(\alpha))$ (again, under the supposition that $\psi$ is in a superposition of the eigenstates of $V(\alpha)$ and thus in the superselection sector(s) of $V(\alpha)$ to begin with).

As we did with $\mathbf{QLI_E}$ before, we can here demonstrate how the $\mathbf{QLI_I}$ semantics for $\nabla\alpha$ do indeed capture our meta-linguistic characterization of indeterminacy for property ascriptions from §2.3. Again, let the propositional variable *p* be a translation of the property ascription <*S* is *F*> in the Kripke-Hilbert model (*H, R, V*) for $\mathbf{QLI_I}$. Then, for any state $\psi$, we have that:

$\nabla p$ is true for $\psi$ in *K* iff $\psi \in ran(V(\nabla p))$
iff $\psi \in ran(V(\neg\Delta p \wedge \neg\Delta\neg p))$
iff $\psi \in ran(V(\neg\Delta p) \wedge V(\neg\Delta\neg p))$
iff $\psi \in ran(V(\neg\Delta p))$ and $\psi \in ran(V(\neg\Delta\neg p))$
iff $\psi \in ran(V(\Delta p)')$ and $\psi \in ran(V(\Delta\neg p)')$
iff (i) $\psi \notin ran(V(\Delta p))$and $\psi \notin ran(V(\Delta\neg p))$, and (ii) $\psi \in dom(V(p))$

iff (i) $\psi \notin ran(V(p))$ and $\psi \notin ran(V(\neg p))$, and (ii) $\psi \in dom(V(p))$
iff (i) $p$ is neither true nor false for ψ, and (ii) $(p \vee \neg p)$ is true for ψ
iff (i) <*S* is *F*> is neither true nor false for ψ, and (ii) it is possible that <*S* is *F*> is true.

Finally, supposing that ψ is the actual state of system *S*, then we have that:

$\nabla p$ is true for ψ in *K* iff (i) <*S* is *F*> is actually neither true nor false, and (ii) it is possible that <*S* is *F*> is true.

This verifies that our truth conditions for $\nabla p$ coincide with our meta-linguistic characterization of indeterminacy for property ascriptions.

## 4.3 Some Logical Equivalences, Entailments, and Other Logical Properties

Just as with $\mathbf{QLI_E}$, among all the logical equivalences for **QL** considered in section §2.2.3, there is only one that does not hold in $\mathbf{QLI_I}$.[17] Again, the exception is one of the two forms of absorption, namely that $(\alpha \vee (\alpha \wedge \beta)) \Leftrightarrow (\alpha \wedge 1_\beta)$. In $\mathbf{QLI_I}$, this equivalence fails exactly when α is a property ascription and β is not. The source of this failure is the way in which the definition of the disjunction interacts with the semantics for the negation when one disjunct is a property ascription and the other is not. To illustrate, let α be a property ascription and let γbe a non-property ascription. Then, for any interpretation function *V* that assigns to α, γ generalized projection operators with the same domain, we have that:

$$V((\alpha \vee \gamma)) = V(\neg(\neg\alpha \wedge \neg\gamma)) = (V(\alpha)^{\perp} \wedge V(\gamma)')',$$

from which it follows that:

$$ran(V((\alpha \vee \gamma))) = ran(V(\alpha)^{\perp}{}') \cup ran(V(\gamma)).$$

Now suppose that $V(\alpha)$ is non-trivial, i.e., not the identity or zero operator on its domain. Then it follows that $ran(V(\alpha)) \subset ran(V(\alpha)^{\perp}{}')$; the latter will also contain all the rays not in either of the ranges of $V(\alpha)$or $V(\alpha)^{\perp}$. We can apply this result to the case of absorption by supposing that

[17] The proofs for these are generally analogous to their proofs in **QL**. The only complication is for disjunctive associativity when not all of the formulas are either property ascriptions or non-property ascriptions. In these cases, one can apply the inset formulas for the interpretation of a disjunction of a property ascription with a non-property ascription.

$\beta$ is a non-property ascription with the same domain as $\alpha$ and setting $\gamma = (\alpha \wedge \beta)$. Then we have that:

$$ran(V(\gamma)) = ran(V(\alpha)) \cap ran(V(\beta)),$$

and hence, given that $V(\alpha)$ is non-trivial, it follows that:

$$ran(V((\alpha \vee (\alpha \wedge \beta)))) = ran(V(\alpha)^{\perp\prime}) \supset ran(V(\alpha)) = ran(V(\alpha \wedge 1_{\beta})).$$

This demonstrates that it is not in general the case that $(\alpha \vee (\alpha \wedge \beta)) \Leftrightarrow (\alpha \wedge 1_{\beta})$.

What about logical equivalences and entailments involving the determinacy and indeterminacy operators? In the following, let $\alpha$ be a formula of $\mathbf{QLI_I}$ and suppose that a Kripke-Hilbert model has been given with interpretation function $V$.

1. $(\Delta\alpha \vee \neg\Delta\alpha) \Leftrightarrow 1_{\alpha}$. This is an application of the excluded middle with the formula $\Delta\alpha$.
2. $\Delta\alpha \Leftrightarrow \alpha$. This follows immediately from the constraints on the interpretation function for any Kripke-Hilbert model.
3. $\Delta^{n}\alpha \Leftrightarrow \Delta^{m}\alpha$ for non-negative integers $n, m$, where for positive $n$, $\Delta^{n}$ abbreviates $n$ concatenated copies of $\Delta$, and $\Delta^{0}\alpha$ just denotes $\alpha$. This follows from the previous logical equivalence by application of mathematical induction.
4. If $\alpha$ is a non-property ascription, then $\neg\Delta\neg\alpha \Leftrightarrow \alpha$. This is because $V(\neg\Delta\neg\alpha) = V(\Delta\neg\alpha)' = V(\neg\alpha)' = V(\alpha)'' = V(\alpha)$. The considerations from the beginning of this section provide an explanation for why this equivalence does not hold when $\alpha$ is a (non-trivial) property ascription.
5. If $\alpha$ is a non-property ascription, then $\neg\nabla\alpha \Leftrightarrow 1_{\alpha}$. This follows from the previous logical equivalence for non-property ascriptions and the definition of the indeterminacy operator. It shows that non-property ascriptions are always either true or false of a state in a Kripke-Hilbert model. Inversely, any witness to indeterminacy, such as those discussed in §2.3, shows that this does not hold for (non-trivial) property ascriptions.

Equivalences (3) and (5) underlie a sense in which there is no higher-order indeterminacy for $\mathbf{QLI_I}$. According to (3), if a proposition holds determinately, then it is determinate that it does so, and vice versa. Similarly, by (5), it is never the case that a proposition asserting something about determinacy (which is what would be expressed by a non-property ascription) is indeterminate. These features of $\mathbf{QLI_I}$ underlie our characterization of indeterminacy from §2.3 as pertaining only to property ascriptions to quantum systems, not necessarily to facts about property ascriptions.

There are also interesting entailments that hold generally of formulas involving (in)determinacy operators but whose converses *do not* hold for (non-trivial) property ascriptions. For instance:

6. $\neg\alpha \vDash \neg\Delta\alpha$. If α is a non-property ascription, then $V(\neg\alpha) = V(\alpha)' = V(\neg\Delta\alpha)$. If α is a property ascription, then $ran(V(\neg\alpha)) = ran(V(\alpha)^{\perp}) \subseteq ran(V(\alpha)') = ran(V(\neg\Delta\alpha))$. If $V(\alpha)$ is a non-trivial property ascription, then the subset inclusion is strict, and so the converse entailment does not hold
7. $\Delta\neg\alpha \vDash \neg\Delta\alpha$. This follows from the previous entailment and the fact that, in general, $V(\Delta\neg\alpha) = V(\neg\alpha)$. However, when α is a property ascription, $V(\neg\Delta\alpha) = V(\alpha)'$, which (when $V(\alpha)$ is non-trivial) is merely a proper subset of $V(\Delta\neg\alpha) = V(\neg\alpha)$, and so the converse entailment does not generally hold.

By contrast, the following entailments, as well as their converses, hold even for any non-trivial non-property ascriptions.

8. $\Delta\alpha \vDash \neg\nabla\alpha$. Since by definition $\neg\nabla\alpha \Leftrightarrow (\Delta\alpha \vee \Delta\neg\alpha)$, $ran(V(\Delta\alpha)) \subseteq ran(\neg\nabla\alpha)$, with the subset inclusion strict just in case $V(\alpha)$ is non-trivial.
9. $\nabla\alpha \vDash \neg\Delta\alpha$. Since by definition $\nabla\alpha \Leftrightarrow (\neg\Delta\alpha \wedge \neg\Delta\neg\alpha)$, $ran(V(\nabla\alpha)) \subseteq ran(\neg\Delta\alpha)$, with the subset inclusion strict just in case $V(\alpha)$ is non-trivial.

The failure of the converse of entailment (6) (and entailment (7), for that matter,) in light of the equivalence (1) illustrates an important feature of $\mathbf{QLI_I}$: in contrast with $\mathbf{QLI_E}$, the semantics of $\mathbf{QLI_I}$ is non-extensional, in the sense that not all entailments are preserved by substitution of extensionally equivalent subformulas. For instance, by equivalence (1), any α is extensionally equivalent with Δα; yet swapping these in the entailment $\neg\alpha \vDash \neg\Delta\alpha$ does not yield a valid entailment of $\mathbf{QLI_I}$. Indeed, the source of this non-extensionality is the way in which the semantics for the negation interact with the determinacy operator in $\mathbf{QLI_I}$. This is why we call $\mathbf{QLI_I}$ the *intensional* extension of **QL** (though, to be clear, by "intensional" we mean nothing more than "non-extensional").

# 5 Williamson's Challenge

In this section we contrast the two approaches developed above by considering the different ways in which they respond to a seminal challenge leveled against the very coherence of indeterminacy. The challenge comes from Williamson (1994), though here we present it in a form that is closer to that given by Barnett (2008). It takes the form of a *reductio* and informally runs as follows, with justifications in brackets:

1. Suppose (for contradiction) that it is indeterminate whether *p*.

2. Then it is not the case that it is determinate that *p*, and it is not the case that it is determinate that not-*p*. [From 1 and the characterization of indeterminacy in terms of determinacy.]
3. But, in general, if something is the case, then it is determinately the case. In particular, if *p*, then it is determinate that *p*; and if not-*p*, then it is determinate that not-*p*.
4. Thus, it is not the case that *p* and it is not the case that not-*p*. [From 2 and 3 by modus tollens and the introduction and elimination rules for the conjunction.]

(4) is a contradiction, and so (1) must be rejected: there is no such thing as indeterminacy—or, at least, it is a concept with no instances.

Any positive account of indeterminacy must be able to say where this argument goes wrong; it must be able to respond to Williamson's Challenge. What is interesting is that $\mathbf{QLI_I}$ and $\mathbf{QLI_E}$ each provide very different diagnoses of where the reasoning goes astray.

To see this, let's begin by translating (1)–(4), as best we can, into a formal language:

1. $\nabla p$
2. $(\neg\Delta p \wedge \neg\Delta\neg p)$ [From 1 and $\nabla p \Leftrightarrow (\neg\Delta p \wedge \neg\Delta\neg p)$.]
3. $((p \rightarrow \Delta p) \wedge (\neg p \rightarrow \Delta\neg p))$
4. $(\neg p \wedge \neg\neg p)$ [From 2 and 3 by modus tollens for $\rightarrow$ and the introduction and elimination rules for the conjunction.]

From our current perspective, however, there are two problems that immediately arise with this presentation of the Challenge. First, neither of the systems $\mathbf{QLI_I}$ and $\mathbf{QLI_E}$, at least as we've presented them so far, provides any semantic definition of a conditional like $\rightarrow$ referenced in (3). And this is significant, for in general there is no consensus regarding how to appropriately capture the material conditional in **QL** (let alone in $\mathbf{QLI_I}$ or $\mathbf{QLI_E}$) (Dalla Chiara and Giuntini 2002, sec. 3). Second, we've provided no proof theory for either $\mathbf{QLI_I}$ or $\mathbf{QLI_E}$, and so we have no precise way of directly verifying (or invalidating) the rules of inference employed in the above argument (e.g., modus tollens for $\rightarrow$).

Our solution to this predicament is to remain focused on the semantic (as opposed to proof-theoretic) relations between formulas. In particular, we will treat both the inferences from one premise to another, as well as the meaning of the conditional $\rightarrow$, as a matter of semantic entailment, which we introduced in §2.2.3. Then we can make the reasoning behind the Challenge fully explicit as follows:

1. Suppose $\nabla p$ is true for some $\psi$ in some model *K*.
2. In general, $\nabla\alpha \vDash (\neg\Delta\alpha \wedge \neg\Delta\neg\alpha)$, for any formula $\alpha$. Thus, $\nabla p \vDash (\neg\Delta p \wedge \neg\Delta\neg p)$, and so $(\neg\Delta p \wedge \neg\Delta\neg p)$ is true for $\psi$ in *K*. [From 1.]
3. In general, $\alpha \vDash \Delta\alpha$, for any formula $\alpha$. Thus, $p \vDash \Delta p$ and $\neg p \vDash \Delta\neg p$.

4. Therefore, both $\neg p$ and $\neg\neg p$ are true for ψ in *K*. [From 2 and 3, by the semantic version of modus tollens for ⊨ and the semantics for the conjunction.]

It is clear that each of $\mathbf{QLI_I}$ and $\mathbf{QLI_E}$ will find *some* flaw in this reasoning. That's because each theory allows for models in which $\nabla p$ is true of some state or another. The interesting question is where, exactly, each theory takes issue with the reasoning in (1)–(4).

Let's begin with $\mathbf{QLI_I}$. According to its semantics, there is no flaw in the above reasoning until we get to the last step, (4). The problem with (4), however, is that the relation of semantic entailment does not generally satisfy the semantic version of modus tollens for $\mathbf{QLI_I}$, and so (4) does not generally follow from (2) and (3). In fact, in general, if $p$ is a (non-trivial) property ascription, then $p \vDash \Delta p$ but it is not the case that $\neg\Delta p \vDash \neg p$. The source of this discrepancy is the different ways in which $\mathbf{QLI_I}$ treats the negation for property ascriptions (like $p$), on one hand, and the negation for non-property ascriptions (like $\Delta p$), on the other.[18] In particular, while in general it is the case that $V(\Delta p) = V(p)$, it is not generally the case that $V(\neg\Delta p) = V(\neg p)$, because $V(\neg\Delta p) = V(\Delta p)'$ whereas $V(\neg p) = V(p)^{\perp}$.

Now consider $\mathbf{QLI_E}$. According to its semantics, and unlike those for $\mathbf{QLI_I}$, the move from (2) and (3) to (4) is valid: the negation is defined extensionally, and so it must be the case that $V(\neg\Delta p) = V(\neg p)$, given that $V(\Delta p) = V(p)$. Instead, the flaw that $\mathbf{QLI_E}$ identifies in the above argument is the first move from (1) to (2). Although it is standard to treat $\nabla p$ as equivalent to $(\neg\Delta p \wedge \neg\Delta\neg p)$ in various discussions of indeterminacy, this equivalence is not validated in $\mathbf{QLI_E}$, as noted in §3.3. Indeed, $(\neg\Delta p \wedge \neg\Delta\neg p)$ is, as Williamson's Challenge suggests, an unsatisfiable formula in $\mathbf{QLI_E}$. It's just that indeterminacy—i.e., $\nabla p$—does not entail any such state of affairs in $\mathbf{QLI_E}$.

It is worth emphasizing that both of these logical theories affirm that, for any formula α, $V(\Delta\alpha) = V(\alpha)$; hence they affirm $\Delta\alpha \Leftrightarrow \alpha$, and so in particular they accept premise (3) in the above argument. In this sense there is no difference, according to these logics, between something's being *determinately* the case and its being the case simpliciter. This is significant, since one way in which Williamson's Challenge is often framed is as asking for an explanation of what the difference could possibly be between determinacy and mere truth. Classical theories of indeterminacy, for instance, would seem to have no way of responding to Williamson's Challenge other than by denying (3) and thus insisting that there is some metaphysical difference between the state of affairs represented by Δα and that represented by α. What we've demonstrated here is that neither $\mathbf{QLI_E}$ nor $\mathbf{QLI_I}$ needs to explain any such difference: each responds to the Challenge by objecting to some other part of the argument.

[18] This semantic version of modus tollens *does* hold when both sides of the entailment relation are either both property ascriptions, or both non-property ascriptions. It only fails to hold when one side is a property ascription and the other is a non-property ascription.

## 6 Conclusions, Comparisons, and Extensions

Let us review what we have accomplished so far. In §2.2, we provided a novel semantics for **QL** using Kripke-Hilbert models that extends standard quantum logic to quantum systems with superselection sectors and property attributions whose corresponding projection operators are defined on only a subset of those sectors. In §2.3, we then supplemented **QL** with a characterization of quantum (in)determinacy using EEL, based on our previous work (Fletcher and Taylor 2021). We then described and explored in §§3 and 4 some of the properties of two distinct extensions to **QL**—$\mathbf{QLI_E}$ and $\mathbf{QLI_I}$, respectively—that implemented this characterization of (in)determinacy through (in)determinacy operators in the formal language itself. Finally, in §5 we showed how each of these extensions provides a distinct answer to Williamson's Challenge to the very idea of indeterminacy generally.

The differences between $\mathbf{QLI_E}$ and $\mathbf{QLI_I}$ naturally lead to the question of which of these two systems, if either, should be preferred as a logic of (in)determinacy. We do not adopt or defend an answer to this question in this essay. Instead, we clarify what would be at stake in the choice between these two logics by comparing their similarities and differences.

Perhaps the most significant difference between $\mathbf{QLI_E}$ and $\mathbf{QLI_I}$ is that $\mathbf{QLI_E}$ retains the extensional semantics for the logical connectives of **QL**, while $\mathbf{QLI_I}$ employs a non-extensional semantics for any formula containing an (in)determinacy operator. In one sense, this makes $\mathbf{QLI_E}$ semantically less of a departure from **QL** and is therefore a reason to prefer for those inclined towards theoretical conservatism. However, this similarity comes with certain comparative costs.

First, $\mathbf{QLI_E}$ does not validate the expected equivalence $\nabla\alpha \Leftrightarrow (\neg\Delta\alpha \wedge \neg\Delta\neg\alpha)$ for all formulas α, which describes how indeterminacy logically bears the same syntactic relationship to determinacy as contingency does to necessity. Instead, the $\mathbf{QLI_E}$ indeterminacy operator has a *sui generis* semantic value not reducible to that assigned to any other type of formula. By contrast, $\mathbf{QLI_I}$ validates this equivalence as expected. This could easily be interpreted as an advantage of $\mathbf{QLI_I}$ over $\mathbf{QLI_E}$. On the other hand, it's worth recalling that the fact that $\mathbf{QLI_E}$ does not validate $\nabla\alpha \Leftrightarrow (\neg\Delta\alpha \wedge \neg\Delta\neg\alpha)$ is not just a pesky, incidental feature of this system; rather, it is essential to how it answers Williamson's challenge, as discussed in §5. From this perspective, the failure of $\nabla\alpha \Leftrightarrow (\neg\Delta\alpha \wedge \neg\Delta\neg\alpha)$ could be considered as representing a fundamentally new way of understanding the logical relationship between determinacy and indeterminacy rather than a flaw of $\mathbf{QLI_E}$.

Second, and connected with the failure of the above equivalence, $\mathbf{QLI_E}$ validates some counterintuitive entailments involving the determinacy operator that $\mathbf{QLI_I}$ does not, but does not validate other intuitive entailments that $\mathbf{QLI_I}$ does. To facilitate their comparison, we have summarized relevant equivalences and entailments in the table below. For each entry, ✓ indicates that the listed equivalence or entailment holds, X indicates that it does not, and *—which appears only under the column for $\mathbf{QLI_I}$—indicates that it holds only for non-property ascriptions.

| **Equivalences** | **$QLI_E$** | **$QLI_I$** |
|---|---|---|
| $\neg\neg\alpha \Leftrightarrow \alpha$ | ✓ | ✓ |
| $(\alpha \wedge \beta) \Leftrightarrow (\beta \wedge \alpha)$<br>$(\alpha \vee \beta) \Leftrightarrow (\beta \vee \alpha)$ | ✓<br>✓ | ✓<br>✓ |
| $(\alpha \wedge (\beta \wedge \gamma)) \Leftrightarrow ((\alpha \wedge \beta)$<br>$(\alpha \vee (\beta \vee \gamma)) \Leftrightarrow ((\alpha \vee \beta)$ | ✓<br>✓ | ✓<br>✓ |
| $\neg(\alpha \wedge \neg\alpha) \Leftrightarrow 1_\alpha$<br>$(\alpha \vee \neg\alpha) \Leftrightarrow 1_\alpha$ | ✓<br>✓ | ✓<br>✓ |
| $(\alpha \wedge (\alpha \vee \beta)) \Leftrightarrow (\alpha \wedge 1_\beta)$<br>$(\alpha \vee (\alpha \wedge \beta)) \Leftrightarrow (\alpha \wedge 1_\beta)$ | ✓<br>X | ✓<br>X |
| $(\Delta\alpha \vee \neg\Delta\alpha) \Leftrightarrow 1_\alpha$<br>$(\Delta\alpha \vee \Delta\neg\alpha) \Leftrightarrow 1_\alpha$ | ✓<br>✓ | ✓<br>X |
| $\Delta\alpha \Leftrightarrow \alpha$<br>$\Delta^n\alpha \Leftrightarrow \Delta^m\alpha$ | ✓<br>✓ | ✓<br>✓ |
| $\nabla\alpha \Leftrightarrow (\neg\Delta\alpha \wedge \neg\Delta\neg\alpha)$<br>$\neg\nabla\alpha \Leftrightarrow 1_\alpha$ | X<br>X | ✓<br>* |

| **Entailments** | **$QLI_E$** | **$QLI_I$** |
|---|---|---|
| $\alpha \vDash \neg\Delta\neg\alpha$<br>$\neg\Delta\neg\alpha \vDash \alpha$ | ✓<br>✓ | ✓<br>* |
| $\neg\alpha \vDash \neg\Delta\alpha$<br>$\neg\Delta\alpha \vDash \neg\alpha$ | ✓<br>✓ | ✓<br>* |
| $\Delta\neg\alpha \vDash \neg\Delta\alpha$<br>$\neg\Delta\alpha \vDash \Delta\neg\alpha$ | ✓<br>✓ | ✓<br>* |
| $\Delta\alpha \vDash \neg\nabla\alpha$<br>$\neg\nabla\alpha \vDash \Delta\alpha$ | ✓<br>X | ✓<br>X |
| $\neg\Delta\alpha \vDash \neg\nabla\alpha$<br>$\neg\nabla\alpha \vDash \neg\Delta\alpha$ | ✓<br>X | *<br>X |
| $\nabla\alpha \vDash \neg\Delta\alpha$<br>$\neg\Delta\alpha \vDash \nabla\alpha$ | X<br>X | ✓<br>X |

In general, **$QLI_E$** validates each of $\neg\Delta\neg\alpha \vDash \alpha$, $\neg\Delta\alpha \vDash \neg\alpha$, $\neg\Delta\alpha \vDash \Delta\neg\alpha$, and $\neg\Delta\alpha \vDash \neg\nabla\alpha$, while **$QLI_I$** does so only for non-property ascriptions. These results are counterintuitive if one's intuition is based in part on the equivalence $\nabla\alpha \Leftrightarrow (\neg\Delta\alpha \wedge \neg\Delta\neg\alpha)$. For instance, to take the first example, denying that $\neg\alpha$ is determinate might seem to allow for two options: either $\alpha$ is true or $\alpha$ is indeterminate. But in **$QLI_E$** only the first of these options can be realized. (Indeterminacy is precluded because $\alpha \vDash \Delta\alpha \vDash \neg\nabla\alpha$ and the excluded middle holds.) In **$QLI_I$**, by contrast, the first option is only mandatory for non-property ascriptions, for which there is no indeterminacy, as described in §4 and formalized in the last equivalency in the above table.

The source of these properties of **$QLI_E$** is the semantic triviality of its determinacy operator: not only is it the case that **$QLI_E$** validates $\Delta\alpha \Leftrightarrow \alpha$ (as **$QLI_I$** does), but it permits the substitution of one formula for the other in the statement of any entailment or equivalence (which **$QLI_I$** does not). This leads **$QLI_E$** to validate $(\Delta\alpha \vee \Delta\neg\alpha) \Leftrightarrow 1_\alpha$, which follows directly from its validation of the excluded middle. One might interpret this equivalence as affirming that for each proposition expressible in $\mathbf{QLI_E}$, either that proposition is determinate or it is not determinate. Yet, surprisingly, this affirmation does not preclude indeterminacy in $\mathbf{QLI_E}$!

This point is related to how $\mathbf{QLI_E}$ does not validate, in general, certain other entailments. For instance, both $\nabla\alpha \vDash \neg\Delta\alpha$ and $\neg\nabla\alpha \Leftrightarrow 1_\alpha$ fail in $\mathbf{QLI_E}$. The failure of the latter in general may not seem to be so problematic in itself. But the former entailment is quite plausible. At least this is true if, again, one thinks that part of what it means to be indeterminate is to not be determinate (an idea which, as we've seen, $\mathbf{QLI_E}$ rejects).

Despite these comparative shortcomings, one should not dismiss the consequences of the non-extensionality of $\mathbf{QLI_I}$. That it clearly precludes higher-order indeterminacy, as discussed in §4.2, is a consequence worth investigating. Moreover, in exchange for validating the reduction of indeterminacy to determinacy via the equivalence $\nabla\alpha \Leftrightarrow (\neg\Delta\alpha \wedge \neg\Delta\neg\alpha)$, $\mathbf{QLI_I}$ only overcomes Williamson's challenge by denying in general the semantic entailments of modus tollens and contraposition. Now, those entailments *do* hold when the formulas in question are either both property ascriptions or both non-property ascriptions. It is only for the mixed case, which is the case that appears essentially in Williamson's argument, that they are denied. We leave it to future investigation whether the non-extensionality and restricted classical entailments of $\mathbf{QLI_I}$ are a greater or lesser cost than the undesirable entailments of $\mathbf{QLI_E}$ and its failure to validate the equivalence $\nabla\alpha \Leftrightarrow (\neg\Delta\alpha \wedge \neg\Delta\neg\alpha)$.

Three further topics suggest themselves for future investigation. First, we have assumed that all property ascriptions are represented by self-adjoint operators, but there has long been motivation within physics to liberalize this requirement, allowing so-called algebras of *effects* or *positive operator-valued measures* (POVMs) that some interpret as representing unsharp or fuzzy properties (Dalla Chiara and Giuntini 2002, secs. 11–12; Busch and Jaeger 2010 resp.). Extending $\mathbf{QLI_E}$ or $\mathbf{QLI_I}$ to such structures may permit the expression of a kind of higher-order indeterminacy, in addition to incorporating a wider and more sophisticated range of the quantum formalism. Second, extending $\mathbf{QLI_E}$ or $\mathbf{QLI_I}$ to first-order languages with identity (cf. Dalla Chiara and Giuntini 2002, sec. 9) might permit expression of the nature and conditions for indeterminate identity—as, say, might obtain regarding systems of multiple electrons—which were some of the original motivations for more formal investigations of quantum indeterminacy (Darby 2010, sec. 2; Bokulich 2014, secs. 2–3 and refs. therein). Third, one might take these quantum logics of indeterminacy as new models for how to formalize indeterminacy outside the context of quantum phenomena. The general strategy of locating (in)determinacy's logical peculiarities in an interaction between the negation and (in)determinacy operator seems to be novel. This strategy may prove fruitful in better understanding the nature of indeterminacy in general.

## References

Akiba, Ken. 2004. "Vagueness in the World." *Noûs* 38 (3): 407–29.

Bacciagaluppi, Guido, and Antony Valentini. 2009. *Quantum Theory at the Crossroads: Reconsidering the 1927 Solvay Conference*. Cambridge: Cambridge University Press.

Barnes, Elizabeth. 2014. "Fundamental Indeterminacy." *Analytic Philosophy* 55: 339–62.

Barnes, Elizabeth, and Ross P Cameron. 2009. "The Open Future: Bivalence, Determinism and Ontology." *Philosophical Studies* 146 (2): 291–309.

Barnes, Elizabeth, and J. Robert G. Williams. 2011. "A Theory of Metaphysical Indeterminacy." In *Oxford Studies in Metaphysics*, edited by Karen Bennett and Dean Zimmerman, 6:103–48. Oxford: Oxford University Press.

Barnett, David. 2008. "Indeterminacy and Incomplete Definitions." *Journal of Philosophy* CV (4): 167–91.

Barrett, Jeffrey A. 1999. *The Quantum Mechanics of Minds and Worlds*. Oxford: Oxford University Press.

Bernstein, Sara. 2016. "Casual and Moral Indeterminacy." *Ratio* 29 (4): 434–47.

Birkhoff, Garrett, and John von Neumann. 1936. "The Logic of Quantum Mechanics." *Annals of Mathematics* 37 (4): 823–43.

Bokulich, Alisa. 2014. "Metaphysical Indeterminacy, Properties, and Quantum Theory." *Res Philosophica* 91 (3): 449–75.

Busch, Paul, and Gregg Jaeger. 2010. "Unsharp Quantum Reality." *Foundations of Physics* 40 (9–10): 1341–67.

Chibeni, Silvio Seno. 2004. "Ontic Vagueness in Microphysics." *Sorites* 15: 29–41.

Dalla Chiara, Maria Luisa, and Roberto Giuntini. 2002. "Quantum Logics." In *Handbook of Philosophical Logic*, edited by Dov Gabbay and Franz Guenthner, 2nd ed., 6:129–228.

Darby, George. 2010. "Quantum Mechanics and Metaphysical Indeterminacy." *Australasian Journal of Philosophy* 88 (2): 227–45.

Dishkant, H. 1972. "Semantics of the Minimal Logic of Quantum Mechanics." *Studia Logica* 30: 23–30.

Dummett, Michael. 1975. "Wang's Paradox." *Synthese* 30 (3–4): 301–24.

Earman, John. 2008. "Superselection Rules for Philosophers." *Erkenntnis* 69 (3): 377–414.

Evans, Gareth. 1978. "Can There Be Vague Objects?" *Analysis* 38 (4): 208.

Field, Hartry. 1973. "Theory Change and the Indeterminacy of Reference." *Journal of Philosophy* 70 (14): 462–81.

Fine, Kit. 1975. "Vagueness, Truth and Logic." *Synthese* 30: 265–300.

Fletcher, Samuel C. and David E. Taylor. 2021. "Quantum Indeterminacy and the Eigenstate-Eigenvalue Link." *Synthese* forthcoming: 1–32. https://dx.doi.org/10.1007/s11229-021-03285-3

Foulis, D., C. Piron, and C. Randall. 1983. "Realism, Operationalism, and Quantum Mechanics." *Foundations of Physics* 13 (8): 813–41.

Gilton, Marian J. R. 2016. "Whence the Eigenstate-Eigenvalue Link?" *Studies in History and Philosophy of Modern Physics* 55: 92–100.

Goswick, Dana. 2019. "Odd Objects: LEM Violations and Indeterminacy." *Erkenntnis* forthcoming: 1–19. https://dx.doi.org/10.1007/s10670-019-00173-8

Greenough, Patrick. 2003. "Vagueness: A Minimal Theory." *Mind* 112 (446): 235–81.

Ismael, Jenann. 2020. "Quantum Mechanics." In *The Stanford Encyclopedia of Philosophy*, Winter 2020. Stanford University: Metaphysics Research Lab. https://plato.stanford.edu/archives/win2020/entries/qm/.

Keefe, Rosanna. 2000. *Theories of Vagueness*. Cambridge: Cambridge University Press.

Kleene, Stephen Cole. 1952. *Introduction to Metamathematics*. New York: Van Nostrand.

Lewis, David K. 1986. *On the Plurality of Worlds*. Oxford: Blackwell.

Lewis, Peter J. 2016. *Quantum Ontology: A Guide to the Metaphysics of Quantum Mechanics*. Oxford: Oxford University Press.

Myrvold, Wayne. 2018. "Philosophical Issues in Quantum Theory." In *The Stanford Encyclopedia of Philosophy*, Fall 2018. Stanford University: Metaphysics Research Lab. https://plato.stanford.edu/archives/fall2018/entries/qt-issues/.

Neumann, John von. 1932. *Mathematische Grundlagen Der Quantenmechanik*. Berlin: Springer.

Parfit, Derek. 1984. *Reasons and Persons*. Oxford: Oxford University Press.

Piziak, R., P. L. Odell, and R. Hahn. 1999. "Constructing Projections on Sums and Intersections." *Computers and Mathematics with Applications* 37: 67–74.

Quine, W. V. 1960. *Word and Object*. Cambridge, MA: MIT Press.

Randall, C. H., and D. J. Foulis. 1983. "Properties and Operational Propositions in Quantum Mechanics." *Foundations of Physics* 13 (8): 843–57.

Rasiowa, Helena, and Roman Sikorski. 1963. *The Mathematics of Metamathematics.* Warsaw: PWN.

Reichenbach, Hans. 1944. *Philosophic Foundations of Quantum Mechanics*. Berkeley: University of California Press.

Ronde, Christian de, Guillem Domenech, and Hector Freytes. n.d. "Quantum Logic in Historical and Philosophical Perspective." In *The Internet Encyclopedia of Philosophy*. Accessed March 25, 2019. https://www.iep.utm.edu/qu-logic/.

Russell, Bertrand. 1923. "Vagueness." *Australasian Journal of Philosophy and Psychology* 1: 84–92.

Skow, Bradford. 2010. "Deep Metaphysical Indeterminacy." *The Philosophical Quarterly* 60 (241): 851–58.

Taylor, David E. 2018. "A Minimal Characterization of Indeterminacy." *Philosophers' Imprint* 18 (5): 1–24.

Torza, Alessandro. 2020. "Quantum Metaphysical Indeterminacy and Worldly Incompleteness." *Synthese* 197 (10): 4251–64.

Wallace, David. 2012a. "Decoherence and Its Role in the Modern Measurement Problem." *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 370 (1975): 4576–93.

———. 2012b. *The Emergent Multiverse*. Oxford: Oxford University Press.

———. 2013. "A Prolegomenon to the Ontology of the Everett Interpretation." In *The Wave Function: Essays on the Metaphysics of Quantum Mechanics*, edited by Alyssa Ney and David Z Albert, 203–22. Oxford: Oxford University Press.

———. 2019. "What Is Orthodox Quantum Mechanics?" In *Philosophers Look at Quantum Mechanics*, edited by Alberto Cordero, 285–312. Synthese Library 406. Cham: Springer.

Wilce, Alexander. 2017. "Quantum Logic and Probability Theory." In *The Stanford Encyclopedia of Philosophy*, Spring 2017. Stanford University: Metaphysics Research Lab. https://plato.stanford.edu/archives/spr2017/entries/qt-quantlog/.

Williams, J. Robert G. 2008. "Multiple Actualities and Ontically Vague Identity." *The Philosophical Quarterly* 58 (230): 134–54.

Williamson, Timothy. 1994. *Vagueness*. London: Routledge.

Wilson, Jessica M. 2013. "A Determinable-Based Account of Metaphysical Indeterminacy." *Inquiry* 56 (4): 359–85.